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# CIRCUIT ANALYSIS OF A-C POWER SYSTEMS

## VOLUME II

By
EDITH CLARKE
PROFESSOR OF ELECTRICAL ENGINEERING
THE UNIVERSITY OF TEXAS

*One of a Series Written in the Interest of*
*the General Electric Advanced*
*Engineering Program*

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# PREFACE

*Circuit Analysis of A-C Power Systems*, Volume II, is a continuation of Volume I. In it, as in Volume I, circuits are analyzed by means of components. Basic equations, relating phase quantities and their symmetrical components and phase quantities and their $\alpha\beta 0$ components, derived and applied in Volume I, are tabulated for ready reference in Chapter I of Volume II.

Insulated cables, various types of transformers and autotransformers, synchronous machines, and induction motors are discussed in detail in Volume II, and their electrical characteristics under normal and abnormal operating conditions determined. Overhead transmission lines, treated in Volume I, are not discussed in Volume II. Curves and charts are given for determining skin effect and proximity effect in circuits of non-magnetic solid and tubular conductors. The effects of open conductors in three-phase circuits supplying ungrounded transformer banks are discussed, and charts are given to show the conditions under which high overvoltage or phase reversal of induction motors or both may occur. Methods are given for determining the impedances seen from relays during power swings, with and without faults.

The characteristic impedances of three-phase synchronous machines are first defined in terms of their direct-axis, quadrature-axis, and zero-sequence components. From these components, the positive-, negative-, and zero-sequence impedances of the machine for use in fundamental-frequency component networks under various operating conditions are derived. Instantaneous phase currents and voltages in terms of their harmonics, which result from unsymmetrical short circuits, are determined by the use of instantaneous $\alpha\beta 0$ components. It is shown that $\alpha\beta 0$ components provide a natural link between phase quantities and their direct-axis, quadrature-axis, and zero-sequence components; they also provide a link between phase quantities and their symmetrical components.

The discussion of transformer banks of single-phase units given in Volume I is extended in Volume II to include banks of four- and five-winding transformers and their equivalent circuits. Included also with their equivalent circuits are autotransformer banks, three-phase transformers and autotransformers, Scott-connected transformers with grounded neutral, zigzag and wye-delta grounding transformers, delta-zigzag and wye-zigzag transformer banks.

In Volume II, as in Volume I, the endeavor has been to present methods of procedure in determining the performance of a-c power systems under normal and abnormal operating conditions. Special attention is also given to the development of equivalent circuits for use in the component networks. Owing to space limitation in a book of this size, all types of equipment and all possible abnormal operating conditions have not been included. It is hoped, however, that the methods of analysis given here can be applied by the operating engineer to other types of equipment and to other abnormal operating conditions which may occur on his system.

The author wishes first of all to express her gratitude to Mr. S. B. Crary, Mr. F. S. Rothe, and Miss Rose Pileggi of the General Electric Company for their material assistance in the final stage of the preparation of this book; to Mr. Crary for his helpful suggestions; to Mr. Rothe for revising the chapter on insulated cables to include present-day cable specifications instead of those in force when the chapter was first written; to Miss Rose Pileggi for typing corrections and changes as well as typing the original manuscript and following the preparation of the figures.

She wishes also to thank all her former associates in the General Electric Company who contributed information and critical reviews of finished chapters; especially Mr. F. H. Buller for his information on insulated cables, Messrs. Charles Concordia and S. B. Crary for their critical reviews of the chapters on synchronous machines and induction motors, Mr. F. S. Rothe for his critical review of the chapters on transformers. Thanks are also due to Professor H. A. Peterson of the University of Wisconsin and Professor Gordon Carter of the University of Virginia who, as former associates in the same office, gave of their time and attention to discussions of material to be included in this book.

Although the greater part of Volume II was written while the author was a member of the Central Station Engineering Department of the General Electric Company, much remained to be done after she joined the Electrical Engineering Department of The University of Texas. The delay in production of Volume II is due in part to the change in point of view from that of an engineer in industry to that of a teacher, and to the attempt to make Volume II a textbook for seniors and graduate students as well as a reference for power system engineers.

EDITH CLARKE

AUSTIN, TEXAS
*September, 1950*

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# CHAPTER I

# INTRODUCTION AND SUMMARY OF EQUATIONS

Methods of circuit analysis are developed in Volume I and applied to the determination of sinusoidal currents and voltages of fundamental frequency in symmetrical and unsymmetrical circuits during balanced and unbalanced conditions. In this volume further applications of these methods are made; also, additional methods of analysis are developed and applied. To make this volume complete in itself for one familiar with methods of analysis based on the use of components, equations developed in Volume I and used repeatedly here are summarized in this chapter. Other equations are repeated as required.

**Notation.** The notation used in this volume is consistent with that used in Volume I. Sinusoidal currents and voltages of fundamental frequency are represented by plane vectors $I$ and $V$ (or $E$), respectively. Positive direction of rotation is counterclockwise. Impedances and admittances are represented by the complex quantities $Z$ and $Y$, respectively. A plane vector $V$ or a complex quantity $Z$ is written without distinguishing mark. Scalar values are inclosed in bars; thus the magnitudes of $V$ and $Z$ are written $|V|$ and $|Z|$. A positive angle $\theta$ is written $\underline{/\theta}$, a negative angle $\theta$ is written $\underline{/-\theta}$ or $\overline{/\theta}$.

In polyphase systems, the phases are indicated by $a, b, c, \cdots$ or $A, B, C, \cdots$. The phase order is understood to be $a, b, c, \cdots$ or $A, B, C, \cdots$. Subscripts $a, b, c, \cdots$ and $A, B, C, \cdots$, used with $V, I, Z$, and $Y$, refer to the phases or to phase conductors.

## METHODS OF COMPONENTS

In analyzing the performance of single-phase or multiphase circuits during unbalanced conditions, calculations are usually simplified if the phase currents and voltages are replaced by a system of components consisting of two groups or sets of components for the single-phase circuit, and as many groups or sets of components as there are phases for the multiphase circuit.

### Three-Phase Circuits

Two systems of components for analyzing three-phase circuits are given in Volume I: symmetrical components and alpha, beta, zero ($\alpha\beta0$) components. The positive-plus-negative, positive-minus-negative, and zero-sequence components of Chapter V, Volume I, are a special case of $\alpha\beta 0$ components.

**Reference for Voltage.** In a grounded system, phase voltages and zero-sequence voltages are referred to ground; in an ungrounded system of negligible capacitance with a neutral conductor and no accidental ground, they are referred to the neutral conductor. Positive- and negative-sequence voltages are referred to neutral; $\alpha$ and $\beta$ voltages are also referred to neutral. The 0 components of the $\alpha\beta 0$ system of components are the same as the zero-sequence components of the symmetrical-component system.

**Reference Phase.** Phase $a$ is arbitrarily selected as reference phase.

**Analysis by Symmetrical Components.** In a three-phase circuit, the phase voltages to ground and their symmetrical components of voltage are related by the equations

$$V_a = V_{a1} + V_{a2} + V_{a0} \tag{1}$$

$$V_b = a^2 V_{a1} + a V_{a2} + V_{a0} \tag{2}$$

$$V_c = a V_{a1} + a^2 V_{a2} + V_{a0} \tag{3}$$

$$V_{a1} = \tfrac{1}{3}(V_a + a V_b + a^2 V_c) \tag{4}$$

$$V_{a2} = \tfrac{1}{3}(V_a + a^2 V_b + a V_c) \tag{5}$$

$$V_{a0} = \tfrac{1}{3}(V_a + V_b + V_c) \tag{6}$$

where subscripts 1, 2, 0 indicate positive-, negative-, and zero-sequence components, respectively; $a = -\tfrac{1}{2} + j\sqrt{3}/2 = 1\underline{/120^\circ}$; $a^2 = -\tfrac{1}{2} - j\sqrt{3}/2 = 1\overline{/120^\circ}$.

If $V$ is replaced by $I$ in the above equations, the corresponding current equations are obtained.

**Sequence Self- and Mutual Impedances.** The positive-, negative-, and zero-sequence components of *voltage drop* in the direction of current flow between the terminals of an *unsymmetrical impedance circuit*, resulting from current flow, are

$$V_{a1} = I_{a1}Z_{11} + I_{a2}Z_{12} + I_{a0}Z_{10} \tag{7}$$

$$V_{a2} = I_{a1}Z_{21} + I_{a2}Z_{22} + I_{a0}Z_{20} \tag{8}$$

$$V_{a0} = I_{a1}Z_{01} + I_{a2}Z_{02} + I_{a0}Z_{00} \tag{9}$$

where the first subscript of $Z$ indicates the sequence of the component of voltage drop produced by the component of current indicated by the second subscript.

In a symmetrical circuit, the mutual impedances between the sequence networks disappear, and [7]–[9] become

$$V_{a1} = I_{a1}Z_{11} = I_{a1}Z_1 \qquad [10]$$

$$V_{a2} = I_{a2}Z_{22} = I_{a2}Z_2 \qquad [11]$$

$$V_{a0} = I_{a0}Z_{00} = I_{a0}Z_0 \qquad [12]$$

$Z_{11}$, $Z_{22}$, and $Z_{00}$ are the positive-, negative-, and zero-sequence self-impedances of a three-phase circuit. In a symmetrical circuit, these impedances are usually written $Z_1$, $Z_2$, and $Z_0$. The mutual impedances between the component networks may be reciprocal ($Z_{12} = Z_{21}$) or non-reciprocal ($Z_{12} \neq Z_{21}$).

In the general case of an unsymmetrical *static circuit*,

$$Z_{11} = Z_{22};\ Z_{10} = Z_{02};\ Z_{20} = Z_{01}$$

But

$$Z_{12} \neq Z_{21};\ Z_{10} \neq Z_{01};\ Z_{20} \neq Z_{02}$$

The mutual impedances are non-reciprocal unless two of the phases have equal self-impedances and equal mutual impedances (including no mutual impedances) with the other phase.

**Analysis by αβ0 Components.** In a three-phase circuit the phase voltages to ground and their $\alpha\beta 0$ components of voltage are related by the equations

$$V_a = V_\alpha + V_0 \qquad [13]$$

$$V_b = -\tfrac{1}{2}V_\alpha + \frac{\sqrt{3}}{2}V_\beta + V_0 \qquad [14]$$

$$V_c = -\tfrac{1}{2}V_\alpha - \frac{\sqrt{3}}{2}V_\beta + V_0 \qquad [15]$$

$$V_\alpha = \frac{2}{3}\left(V_a - \frac{V_b + V_c}{2}\right) \qquad [16]$$

$$V_\beta = \frac{1}{\sqrt{3}}(V_b - V_c) \qquad [17]$$

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \qquad [18]$$

where subscripts $\alpha$, $\beta$, 0 indicate $\alpha$, $\beta$, and 0 components, respectively.

If $V$ is replaced by $I$ in the above equations, the corresponding current equations are obtained.

**αβ0 Self- and Mutual Impedances.** The $\alpha$, $\beta$, and 0 components of *voltage drop* in the direction of current flow between the terminals of an *unsymmetrical impedance circuit*, resulting from current flow, are

$$V_\alpha = I_\alpha Z_{\alpha\alpha} + I_\beta Z_{\alpha\beta} + I_0 Z_{\alpha 0} \quad [19]$$

$$V_\beta = I_\alpha Z_{\beta\alpha} + I_\beta Z_{\beta\beta} + I_0 Z_{\beta 0} \quad [20]$$

$$V_0 = I_\alpha Z_{0\alpha} + I_\beta Z_{0\beta} + I_0 Z_{00} \quad [21]$$

where the first subscript of $Z$ indicates the component of voltage drop produced by the component of current indicated by the second subscript.

In a symmetrical static circuit, the mutual impedances disappear and [19]–[21] become

$$V_\alpha = I_\alpha Z_{\alpha\alpha} \quad \text{or} \quad I_\alpha Z_\alpha \quad [22]$$

$$V_\beta = I_\beta Z_{\beta\beta} \quad \text{or} \quad I_\beta Z_\beta \quad [23]$$

$$V_0 = I_0 Z_{00} \quad \text{or} \quad I_0 Z_0 \quad [24]$$

where $Z_{\alpha\alpha}$, $Z_{\beta\beta}$, $Z_{00}$ are the $\alpha$, $\beta$, and 0 self-impedances, which in a symmetrical circuit may be written $Z_\alpha$, $Z_\beta$, and $Z_0$.

In an unsymmetrical static circuit, in the general case, $Z_{\alpha\alpha} \neq Z_{\beta\beta}$, but $Z_{\alpha\beta} = Z_{\beta\alpha}$, and by a simple modification of the 0 network the mutual impedances between the $\alpha$ and 0 and between the $\beta$ and 0 networks can be made reciprocal (Volume I, page 318). Unsymmetrical static circuits, in the general case, are more simply analyzed by $\alpha\beta 0$ components than by symmetrical components.

**Relations between αβ0 Components and Symmetrical Components in a Three-Phase System.**

$$V_\alpha = V_{a1} + V_{a2} \quad [25]$$

$$V_\beta = -j(V_{a1} - V_{a2}) \quad [26]$$

$$I_\alpha = I_{a1} + I_{a2} \quad [27]$$

$$I_\beta = -j(I_{a1} - I_{a2}) \quad [28]$$

$$V_{a1} = \tfrac{1}{2}(V_\alpha + jV_\beta) \quad [29]$$

$$V_{a2} = \tfrac{1}{2}(V_\alpha - jV_\beta) \quad [30]$$

$$I_{a1} = \tfrac{1}{2}(I_\alpha + jI_\beta) \quad [31]$$

$$I_{a2} = \tfrac{1}{2}(I_\alpha - jI_\beta) \quad [32]$$

$$V_0 = V_{a0}; \quad I_0 = I_{a0}$$

In an ***unsymmetrical three-phase static circuit,***

$$Z_{11} = Z_{22}; \qquad Z_{10} = Z_{02}; \qquad Z_{20} = Z_{01}$$

$$Z_{\alpha\alpha} = Z_{11} + \tfrac{1}{2}(Z_{21} + Z_{12}) \tag{33}$$

$$Z_{\beta\beta} = Z_{11} - \tfrac{1}{2}(Z_{21} + Z_{12}) \tag{34}$$

$$Z_{\alpha\beta} = Z_{\beta\alpha} = j\tfrac{1}{2}(Z_{21} - Z_{12}) \tag{35}$$

$$Z_{\alpha 0} = 2Z_{0\alpha} = Z_{10} + Z_{20} \tag{36}$$

$$Z_{\beta 0} = 2Z_{0\beta} = -j(Z_{10} - Z_{20}) \tag{37}$$

$$Z_{11} = Z_{22} = \tfrac{1}{2}(Z_{\alpha\alpha} + Z_{\beta\beta}) \tag{38}$$

$$Z_{12} = \tfrac{1}{2}(Z_{\alpha\alpha} - Z_{\beta\beta}) + jZ_{\alpha\beta} \tag{39}$$

$$Z_{21} = \tfrac{1}{2}(Z_{\alpha\alpha} - Z_{\beta\beta}) - jZ_{\alpha\beta} \tag{40}$$

$$Z_{10} = Z_{02} = Z_{0\alpha} + jZ_{0\beta} \tag{41}$$

$$Z_{01} = Z_{20} = Z_{0\alpha} - jZ_{0\beta} \tag{42}$$

$Z_{00}$, the zero-sequence self-impedance, is the same in both systems.

## Single-Phase and Two-Phase Circuits

In a single-phase or a two-phase circuit, analyzed by means of positive- and zero-sequence symmetrical components (Volume I, pages 287, 297) where phase voltages and zero-sequence voltages are referred to ground or to a neutral conductor, and positive-sequence voltages are referred to neutral,

$$V_a = V_{a1} + V_{a0} \tag{43}$$

$$V_b = -V_{a1} + V_{a0} \tag{44}$$

$$V_{a1} = \tfrac{1}{2}(V_a - V_b) \tag{45}$$

$$V_{a0} = \tfrac{1}{2}(V_a + V_b) \tag{46}$$

If $V$ is replaced by $I$ in the above equations, the corresponding current equations are obtained.

**Positive- and Zero-Sequence Self- and Mutual Impedances.** The positive- and zero-sequence components of *voltage drop* in the direction of current flow between the terminals of an *unsymmetrical impedance circuit,* resulting from current flow, are

$$V_{a1} = I_{a1}Z_{11} + I_{a0}Z_{10} \tag{47}$$

$$V_{a0} = I_{a1}Z_{01} + I_{a0}Z_{00} \tag{48}$$

where the first subscript of $Z$ indicates the sequence of the component of voltage drop produced by the component of current indicated by the second subscript.

If the conductors $a$ and $b$ are identical, $Z_{10} = Z_{01}$. If, in addition, they are equidistant from ground and from the neutral conductor of a three-wire circuit, $Z_{01} = Z_{10} = 0$, and [47] and [48] become

$$V_{a1} = I_{a1}Z_{11} = I_{a1}Z_1 \tag{49}$$

$$V_{a0} = I_{a0}Z_{00} = I_{a0}Z_0 \tag{50}$$

where $Z_{11}$ and $Z_{00}$ are the positive- and zero-sequence self-impedances, which in a symmetrical circuit may be written $Z_1$ and $Z_0$.

## PARALLEL CYLINDRICAL CONDUCTORS IN NON-MAGNETIC MEDIA

**Self- and Mutual Impedances of Conductors.** In the following equations, uniform current distribution over the circular cross sections of the conductors is assumed. Dimensions are in centimeters; $\ell$ is length; $d$ or $D$ is diameter; $s$ is spacing between axes of conductors; $\ell$ is large relative to $d$ and $s$ (Volume I, page 364).

The total *self-inductance* $L$ of a conductor is the sum of its external inductance $L_e$ and its internal inductance $L_i$:

$$L = L_e + L_i \tag{51}$$

The *external self-inductance* $L_e$ of a conductor of outside diameter $d$ and length $\ell$ is

$$L_e = 2\ell\left(\log_e \frac{4\ell}{d} - 1\right) \text{ abhenries} \tag{52}$$

The *internal self-inductance* $L_i$ of a *solid non-magnetic* conductor is

$$L_i = \tfrac{1}{2} \text{ abhenry per cm} \tag{53}$$

**Internal Reactance.** $x_i$ of a solid non-magnetic conductor at constant frequency $f$, under the assumption of uniform current distribution, is

$$x_i = 2\pi f L_i = \frac{f}{60}(0.00575) \text{ ohms per 1000 feet} \tag{54}$$

The internal reactances of conductors of concentric stranding are given in Table I, Chapter II, of this volume.

The internal reactance of a non-magnetic *hollow* conductor of inner and outer diameters $d_i$ and $d_o$, respectively, in ohms per 1000 feet is

$$x_i = \frac{f}{60}\left[0.0529 \frac{d_i^4}{(d_o^2 - d_i^2)^2} \log_{10} \frac{d_o}{d_i} + 0.00575 \frac{d_o^2 - 3d_i^2}{d_o^2 - d_i^2}\right] \tag{55}$$

$x_i$ versus $d_i/d_o$, calculated from [55] at 60 cps (cycles per second) is given in Fig. 4, page 524 of Volume I, in ohms per mile; in Fig. 1 of Chapter II of this volume, it is given in ohms per 1000 feet. Under the assumption of uniform current distribution, $x_i$ varies directly with frequency.

**Mutual Inductance.** The mutual inductance $M_{ab}$ between two parallel non-concentric conductors $a$ and $b$ is

$$M_{ab} = 2\ell\left(\log_e \frac{2\ell}{s} - 1\right) \text{abhenries} \qquad [56]$$

The mutual inductance $M_{aw}$ between a tubular conductor $w$ with outside and inside diameters $D_o$ and $D_i$, respectively, and a concentric inner conductor $a$ is

$$M_{aw} = 2\ell\left(\log_e \frac{4\ell}{D_o} - \frac{D_i^2}{D_o^2 - D_i^2}\log_e \frac{D_o}{D_i} - \frac{1}{2}\right) \text{abhenries} \qquad [57]$$

As an approximation when $D_i/D_o$ approaches unity, the total inductance $L_{ww}$ of the tubular conductor $w$ and the mutual inductance $M_{aw}$ between $a$ and $w$ can be assumed equal and calculated from [52] with $d$ replaced by the average diameter $\frac{1}{2}(D_o + D_i)$ of $w$. Then

$$M_{aw} = L_{ww} = 2\ell\left(\log_e \frac{4\ell}{\frac{1}{2}(D_o + D_i)} - 1\right) \text{abhenries} \qquad [58]$$

**Sequence Self- and Mutual Impedances of a Three-Wire Three-Phase Circuit.** (Volume I, page 369.) When the phase conductors $a$, $b$, and $c$ are identical cylindrical conductors and the effect of the earth on positive- and negative-sequence impedances is negligible, the sequence self- and mutual impedances (except $Z_{00}$) in ohms per 1000 feet, assuming uniform current distribution, are

$$Z_{11} = Z_{22} = r + j\left(0.0529\frac{f}{60}\log_{10}\frac{2s}{d} + x_i\right) \qquad [59]$$

$$Z_{12} = j0.0353\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} + j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right) \qquad [60]$$

$$Z_{21} = j0.0353\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right) \qquad [61]$$

$$Z_{10} = Z_{02} = -\tfrac{1}{2}Z_{21} \qquad [62]$$

$$Z_{20} = Z_{01} = -\tfrac{1}{2}Z_{12} \qquad [63]$$

where $r$ is resistance and $x_i$ internal reactance of one conductor in ohms per 1000 feet; $d$ is outside diameter of conductors; $s$ is spacing between axes of conductors indicated by the subscripts; $s$ without subscripts is the geometric mean distance between axes of conductors:

$$s = \sqrt[3]{s_{ab}s_{ac}s_{bc}} \tag{64}$$

When $s_{ab} = s_{ac} = s_{bc}$, the circuit is symmetrical and the mutual impedances between the sequence networks disappear. In this case, [7]–[9] reduce to [10]–[12]. When $s_{ac} = s_{ab}$, the mutual impedances in [7]–[9] are reciprocal.

**Sequence Self- and Mutual Impedances of a Single-Phase or Two-Phase Circuit.** In a single-phase or two-phase circuit of identical phase conductors, $a$ and $b$, analyzed by positive- and zero-sequence symmetrical components (Volume I, page 386), the positive-sequence self-impedance $Z_{11}$ is the same as $Z_{11} = Z_{22}$ for the three-phase circuit given by [59], $s$ being the spacing between the axes of the two conductors. When zero-sequence currents return in an ungrounded neutral conductor $n$,

$$Z_{10} = Z_{01} = j0.0529\frac{f}{60}\log_{10}\frac{s_{an}}{s_{bn}} \tag{65}$$

If $s_{an} = s_{bn}$, $Z_{10} = Z_{01} = 0$ and [47] and [48] reduce to [49] and [50].

# CHAPTER II

## IMPEDANCES OF ELECTRIC CIRCUITS

The characteristics of an electric circuit are *resistance*, *inductance*, *capacitance*, and *leakance*. These characteristics depend upon the dimensions and materials of the circuit. Resistance and inductance determine series impedance; leakance and capacitance determine shunt admittance.

Leakance in many circuits during normal operating conditions is negligible, or actually zero. An exception occurs in insulated cables, where leakance during normal operation may have an appreciable effect in limiting the current rating of the cable. In circuits where leakance is negligible during normal operation, it may become of importance during overvoltages; for example, corona must be considered in overhead transmission lines if the voltage between conductors or to ground exceeds corona-starting voltage for the line (Volume 1, page 528).

Resistance, inductance, and capacitance are present in all circuits but, depending upon the circuit and the problem under consideration, one or more of these quantities may be unimportant in its effects upon final results. Inductive impedances and capacitive admittances of overhead transmission lines are treated in Chapters XI and XII of Volume I. Capacitance and leakance of insulated cables are discussed in Chapter III of this volume. This chapter deals with resistances and reactances of electric circuits in general and the factors which affect their magnitudes.

### TRANSMISSION CIRCUITS

The conductors of transmission circuits are bare or insulated. When space or safety is important, the conductors are insulated and placed closer together than is possible with bare conductors.

**Overhead Transmission Lines.** Except in aerial cables, the conductors of overhead transmission circuits are bare. The distances between conductors is sufficiently large for proximity effect upon impedance to be negligible. Skin effect, however, may be appreciable.

In Appendix B, Volume I, tables of d-c and a-c resistances at 25°C, conductor equivalent geometric mean radii, and internal reactances of conductors used in overhead transmission lines are given for frequencies of 25, 50, and 60 cps in which skin effect is included. The tables are for hard-drawn solid, stranded and hollow copper conductors, aluminum cable, steel reinforced (A.C.S.R.), Copperweld conductors, and steel and iron wires.

For *magnetic or partly magnetic conductors*, skin effect depends upon current as well as upon material, size, and frequency. The values in Tables IV–VI of Appendix B, Volume I, given at several values of current or current per square inch, were tabulated from data furnished by manufacturers.

For *non-magnetic conductors*, skin effect is independent of current. Skin effect and the effect of spiraling were taken into account (as explained later) in determining the a-c resistances and internal reactances of the hard-drawn copper conductors of Appendix B, Volume I.

**Insulated Cables.** At the present time annealed copper is used for the conductors of insulated cables. Aluminum has also been considered for use in cables, the disadvantage being the increase in the diameter of the conductor for the same resistance. This increases the overall diameter of the cable and the amount of insulation and lead sheath. Because of the use of copper in electrical equipment, and also of its use as a standard of reference for other metals, a brief discussion of its properties will be given.

## COPPER CONDUCTORS

The important characteristics of copper used for conductors are conductivity, tensile strength, softening temperature, and resistance to change in internal structure. These characteristics are greatly influenced by composition and manufacturing processes. Because of this and the many and varied uses of copper conductors, there are many classes of copper.

Well-reduced and refined copper is called *tough-pitch* copper. Oxygen is present in tough-pitch copper. Copper without oxygen is called *oxygen-free* copper. In hydrogen-cooled machines, any oxygen present in the copper tends to unite with the hydrogen at high temperatures and form water vapor. This union may result in hydrogen embrittlement. To prevent it, oxygen-free copper is used. In both tough-pitch and oxygen-free copper small amounts (about 0.01%) of other metallic elements may be present. Although copper of high purity is essential for high conductivity, spectroscopically pure copper is not required; nor is it even desirable at high temperatures, as there are certain metallic impurities which only slightly reduce the conductivity while greatly increasing the softening temperature.[1] (For further information on copper metallurgy see reference 1 and its bibliography.)

Rolling and drawing harden copper. Subsequent annealing (heating) softens it. Annealed copper has higher conductivity but less tensile strength than hard-drawn copper. Hard-drawn copper is used in overhead transmission lines because of its greater tensile strength. Annealed copper is used in insulated cables and in electrical equipment where high tensile strength is not required; therefore advantage can be taken of its high conductivity. Because of the importance of copper in industry, an international copper standard has been adopted.

### International Standard of Resistance for Copper

The present standard[2] for annealed copper was adopted by the International Electrotechnical Commission in 1913. The following rules of this Commission have been adopted by the American Institute of Electrical Engineers:[3]

(*a*) At a temperature of 20°C, the resistance of a wire of standard annealed copper one meter in length and of a uniform section of one square millimeter is 1/58 ohm = 0.017241 . . . ohm.

(*b*) At a temperature of 20°C, the density of standard annealed copper is 8.89 grams per cubic centimeter.

(*c*) At a temperature of 20°C, the " constant mass " temperature coefficient of resistance of standard annealed copper, measured between two potential points rigidly fixed to the wire, is 0.00393 = 1/254.45 . . . per degree Centigrade.

(*d*) As a consequence, it follows from (*a*) and (*b*) that at a temperature of 20°C the resistance of a wire of standard annealed copper of uniform section, one meter in length and weighing one gram, is (1/58) × 8.89 = 0.15328 . . . ohm.

Resistivity defined under (*a*) is volume resistivity; resistivity defined under (*d*) is mass resistivity. These two values of resistivities are related through density. Values of the resistivity of standard annealed copper at 20°C in various units of " volume resistivity " and " mass resistivity," written in the customary manner, are given in Table I.

TABLE I

RESISTIVITY $\rho$ OF STANDARD ANNEALED COPPER AT 20°C

$\rho$ (volume resistivity) = 0.017241 ohm (meter, $mm^2$)
= 1.7241 microhms (cm cube) or microhms-cm
= 1724.1 abohms (cm cube) or abohms-cm
= 0.67879 microhm (inch cube) or microhm-inch
= 10.371 ohms (mil, foot)
$\rho$ (mass resistivity) = 0.15328 ohm (meter, gram)
= 875.20 ohms (mile, pound)

The *conductivity of metals* at 20°C is expressed in per cent or per unit of the conductivity of the annealed copper standard at 20°C. The conductivity of the hard-drawn copper used in overhead transmission lines is approximately 97.3% or 0.973 per unit; that of the annealed copper used in insulated cables is approximately 100% or unity.

When $n$, the per unit conductivity of a metal, is given, its resistivity at 20°C is $1/n$ times the resistivity of standard annealed copper at 20°C. Unless otherwise specified, *the given per cent or per unit conductivity of a metal is its conductivity at* 20°C *in per cent or per unit of the conductivity of standard annealed copper at* 20°C. This simple relation between resistivity and per unit conductivity holds only at 20°C.

The *constant-mass temperature coefficient of resistance* of standard copper at 20°C from (*c*) is 0.00393 per centigrade degree. For any sample of copper at $t$°C, the temperature coefficient[2] $\alpha_t$ is

$$\alpha_t = \frac{n(0.00393)}{1 + n(0.00393)(t - 20)} \tag{1}$$

where $n$ is per unit conductivity (per cent conductivity divided by 100).

For *standard copper*, $n = 1$. From [1],

$$\text{at } t = 0°\text{C}, \quad \alpha_t = \alpha_0 = 0.00427$$

$$\text{at } t = 25°\text{C}, \quad \alpha_t = \alpha_{25} = 0.00385$$

For *hard-drawn copper*, $n = 0.973$. From [1],

$$\text{at } t = 20°\text{C}, \quad \alpha_t = \alpha_{20} = 0.00382$$

$$\text{at } t = 25°\text{C}, \quad \alpha_t = \alpha_{25} = 0.00375$$

When the *d-c resistance* $R_{t1}$ of a conductor of constant mass at a specified temperature $t_1$ in degrees centigrade is known, and also the constant-mass temperature coefficient of resistance $\alpha_{t1}$ of the material at that temperature, the d-c resistance $R_t$ at any temperature $t$°C can be calculated from the equation

$$R_t = R_{t1}[1 + \alpha_{t1}(t - t_1)] \tag{2}$$

**Coefficient of Thermal Expansion.**[2] The linear thermal expansion of copper is not a linear function of temperature but, within the temperature range between −40°C and 300°C, it is well expressed by a quadratic equation. Within the temperature range encountered in power conductor under normal operation, $\gamma$, the coefficient of linear thermal expansion per centigrade degree to two significant figures, is

$$\gamma = 0.000017$$

The length $\ell_t$ and the cross section $A_t$ of a copper conductor at temperature $t$°C are approximately

$$\ell_t = \ell_{20}[1 + 0.000017(t - 20)] \qquad [3]$$

$$A_t = A_{20}[1 + 0.000017(t - 20)]^2$$

$$= A_{20}[1 + 0.000034(t - 20)], \quad \text{since } \gamma \text{ is very small} \qquad [4]$$

where $\ell_{20}$ and $A_{20}$ are the length and cross section at 20°C.

**Resistivity-Temperature Constant.**[2] The change of resistivity of copper per centigrade degree is a constant, independent of the temperature of reference and of the sample of copper; this constant is 0.00681 microhm-cm or 0.000597 ohm (meter, gram).

*Volume* and *mass* resistivities $\rho_v$ and $\rho_m$, respectively, at any temperature $t$ are

$$\rho_{vt} = \rho_{v-20} + 0.00681(t - 20) \text{ microhms (cm cube)} \qquad [5]$$

$$\rho_{mt} = \rho_{m-20} + 0.000597(t - 20) \text{ ohm (meter, gram)} \qquad [6]$$

where $\rho_{v-20}$ and $\rho_{m-20}$ are volume and mass resistivities at 20°C in microhms (cm cube) and ohm (meter, gram), respectively, of the sample of copper.

**Equations in Terms of Resistivity.** When equations involving conductors of constant mass are expressed in terms of $\rho$ the resistivity of the material in abohms (cm cube) and this resistivity varies with temperature and is accompanied by changes in dimensions with temperature, it is convenient to replace $\rho$ by its value in terms of $R_{dc}$, where $R_{dc}$ is d-c resistance in ohms per 1000 feet at the given temperature. This is illustrated in [23]–[25] and [29]–[31].

## D-C RESISTANCE

When direct current flows in a long homogeneous conductor of uniform cross section, it is uniformly distributed over the cross section of the conductor. The d-c resistance $R$ of the conductor at any temperature $t$°C is

$$R = \frac{\rho_v \ell}{A} \qquad [7]$$

or

$$R = \frac{\rho_m \ell^2}{m} \qquad [8]$$

where $\ell$ is length, $A$ is area of cross section of material, $m$ is mass, $\rho_v$ and $\rho_m$ are volume and mass resistivity, respectively, all at temperature $t$°C.

*The d-c resistance at 20°C of a conductor of constant mass and* dimensions given at 20°C can be determined from *[7]; then,* $R_{20}$ at 20°C being known, the resistance at any temperature $t$°C can be obtained from [2]. Or the following equation can be used directly:

$$R_t = \frac{\rho_{20} l_{20}}{A_{20}} [1 + \alpha_{20}(t - 20)] \tag{9}$$

where $R_t$ is d-c resistance at $t$°C, $\rho$ is volume resistivity, and the subscript 20 indicates that $\rho$, $l$, $A$, and $\alpha$ are at 20°C.

Equation [9] is directly applicable to a homogeneous conductor of any material with uniform cross section which is long enough for end effects to be negligible. To apply [9], the resistivity $\rho_{20}$ and the constant-mass temperature coefficient of resistance $\alpha_{20}$ at 20°C must be known. Resistivities and constant-mass temperature coefficients of resistance of the various metals used in electrical circuits are listed in handbooks.

**Conductors of Circular Cross Section.** These are called round conductors; they include solid, tubular, stranded, and stranded annular conductors. $A$, the area of cross section of material of a round conductor, is usually given in circular mils. Its diameter $d$ is usually given in inches or mils, where 1 inch = 1000 mils.

In a *solid conductor* of diameter $d$ and cross section $A$,

$$A \text{ (cir mils)} = (d \text{ mils})^2 \tag{10}$$

$$A \text{ (sq in.)} = \frac{\pi}{4} A \text{ (cir mils) } 10^{-6} \tag{11}$$

$$d \text{ (in.)} = \sqrt{\frac{4}{\pi} A \text{ (sq in.)}} = 10^{-3}\sqrt{A \text{ (cir mils)}} \tag{12}$$

$$d \text{ (cm)} = \sqrt{\frac{4}{\pi} A \text{ (sq cm)}} = 2.54 \times 10^{-3}\sqrt{A \text{ (cir mils)}} \tag{13}$$

In a *tubular conductor* (non-stranded) of outside diameter $d$ and thickness $t$,

$$A\text{(sq in.)} = \frac{\pi}{4}[d \text{ (in.)}]^2 \left[\frac{4t}{d}\left(1 - \frac{t}{d}\right)\right] \tag{14}$$

$$A \text{ (cir mils) } 10^{-6} = [d \text{ (in.)}]^2 \left[\frac{4t}{d}\left(1 - \frac{t}{d}\right)\right] \tag{15}$$

$$A \text{ (sq cm)} = \frac{\pi}{4}[d \text{ (cm)}]^2 \left[\frac{4t}{d}\left(1 - \frac{t}{d}\right)\right] \tag{16}$$

where $t$ and $d$ in $t/d$ are expressed in the same unit. A solid conductor is a special case of a tubular conductor in which $t/d = 0.5$. When this substitution is made in [15] or [16], $d$ is given by [12] or [13].

**Stranded Conductors.** A concentric-lay cable, or a cable of concentric stranding, is a conductor in which a central straight bare wire is surrounded by helical layers of bare wires. All wires are of the same diameter, and alternate layers are spiraled in opposite directions. In the first layer about the central wire, there are 6 wires; in the second layer, 12; then 18, 24, 30, and so on. The number of layers determines the number of wires in the conductor. Concentric-lay cables have 7, 19, 37, 61, 91, 127, or 169 strands. Table III, Appendix A, gives number of wires, outside diameters, and d-c resistances of bare concentric-lay cables of standard annealed copper. The conductors commonly used in insulated power cables are shown in this table under " standard strands "; the conductors listed under " flexible strands " have a larger number of wires to give greater flexibility.

Conductors of flexible stranding (not to be confused with flexible *concentric* stranding discussed above) are conductors of special flexibility, made up of a large number of wires of a definite gauge size.[2] (A.I.E.E. Standards, No. 30, 1925, gives information on the sizes and number of wires used in *flexible cables* or cables of flexible stranding.)

**Stranded Annular Conductors.** Table IV, Appendix A, gives information on the stranded annular conductors with rope centers used in insulated power cables.

**D-C Resistances of Round Conductors of Standard Annealed Copper.** When cross section and length are known at 20°C, the d-c resistance $R_t$ at temperature $t$°C, from [9], in ohms per 1000 feet is

$$R_t = \frac{10.371 \times 10^3}{A \text{ (cir mils)}} [1 + 0.00393(t - 20)] \qquad [17]$$

Equation [17] is directly applicable to solid or tubular conductors of standard annealed copper. For conductors of *concentric stranding* up to 2,000,000 cir mils, $R_t$ calculated from [17] is increased 2% to allow for the greater length of the individual wires than of the conductor itself because of spiraling.[2] For stranded annular conductors, this increase ranges from 3% to 5%.

**Copper Wire Tables.** The resistances of the conductors of standard annealed copper given in Appendix A, Tables II–IV, are for conductors of constant mass with dimensions given at 20°C. The given diameters and cross sections of the conductor are assumed to be exact at 20°C and to increase or decrease with temperature. The length is understood to be 1000 feet at 20°C and to vary with temperature. As the mass remains constant, the constant-mass temperature coefficient of resistance at 20°C of standard annealed copper is used to determine d-c resistances at temperatures other than 20°C, without the necessity of considering change of dimensions with temperature.

In tables in which length and cross section apply at each temperature listed, the conductor has constant volume but varying mass. (There is a conductor of different mass at each temperature.) To prepare a table of this type, volume resistivity $\rho_v$ at each listed temperature can be calculated from [5] and substituted in [7] to obtain d-c resistance, where $\ell$ and $A$ are constant length and cross section at all temperatures.

In power-system problems it is usually unimportant whether recorded conductor dimensions are given at 20°C or at a higher or lower operating temperature, because the change in dimensions with temperature is but slight throughout the range of temperature encountered in power systems during normal operation. See [3] and [4]. The two types of tables are discussed here to explain minor differences in d-c resistances given in tables for conductors of constant mass with dimensions at 20°C and in tables of constant dimensions at all temperatures.

**Example 1.** (*a*) Calculate the d-c resistance at 70°C in ohms per 1000 feet of a standard annealed copper conductor of 2,000,000 cir mils, standard concentric stranding. (*b*) What would be the d-c resistance of a similar conductor of hard-drawn copper of 97.3% conductivity? (*c*) Of aluminum of 61% conductivity, steel-reinforced, with $\alpha_{20} = 0.00403$?

*Solution.* (*a*) Multiplying [17] by 1.02 to allow for the increase in length of the strands due to spiraling,

$$R \text{ (at 70°C)} = 1.02 \frac{10.371 \times 10^3}{2 \times 10^6} [1 + 0.00393(70 - 20)]$$

$$= 0.00633 \text{ ohm per 1000 feet}$$

Equation [9], multiplied by 1.02 to allow for stranding, will be used for (*b*) and (*c*).

(*b*) The resistivity of copper of 97.3% conductivity at 20°C is

$$\rho_{20} = \frac{10.371}{0.973} = 10.66 \text{ ohms (mil, foot)}$$

From [1],

$$\alpha_{20} \text{ (at 20°C)} = 0.00382$$

$$R \text{ (at 70°C)} = 1.02 \times \frac{10.66 \times 10^3}{2 \times 10^6} [1 + 0.00382(70 - 20)]$$

$$= 0.00647 \text{ ohm per 1000 feet}$$

(*c*) Neglecting the presence of the steel,

$$\rho_{20} = \frac{10.371}{0.61} = 17.002 \text{ ohms (mil, foot)}$$

$$\alpha_{20} \text{ (at 20°C)} = 0.00403 \text{ (given)}$$

$$R \text{ (at 70°C)} = 1.02 \frac{17.00 \times 10^3}{2 \times 10^6} [1 + 0.00403(70 - 20)]$$

$$= 0.0104 \text{ ohm per 1000 feet}$$

## IMPEDANCES OF A-C CIRCUITS IN NON-MAGNETIC MEDIA

In an a-c circuit, resistance changes with temperature just as in a d-c circuit. In addition, resistance and reactance may be influenced by *non-uniform current distribution* over the cross section of the conductors.

### Skin Effect and Proximity Effect in Round Conductors

*Skin effect* is the phenomenon of non-uniform current distribution over the cross section of a conductor caused by variation of the current in the conductor itself. *Proximity effect* is the phenomenon of non-uniform current distribution over the cross section of a conductor caused by variation of current in a neighboring conductor (or conductors). The non-uniform current distribution due to skin effect is symmetrical about the axis of symmetry of a round conductor; that due to proximity effect is unsymmetrical. Skin effect increases the effective a-c resistance of a conductor and decreases its *internal inductance*. It does not affect inductance external to the conductor. Skin effect is independent of circuit configuration and the sequence of the current. It depends upon frequency and the dimensions and materials of the conductor. Proximity effect depends upon flux distribution both inside and outside the conductors. In addition to frequency and conductor dimensions and materials, it is influenced by circuit configuration and the relative magnitudes and phase relations between the currents in the conductors. Proximity effect is not the same, therefore, with zero-sequence currents in the conductors as with currents of positive sequence. With positive-sequence currents only in the conductors, it is different for a single-phase circuit and a symmetrical three-phase circuit with the same conductors and spacing between conductors. *The method of symmetrical components cannot be applied to determine proximity effect during unbalanced system conditions.*

By the method of symmetrical components, phase currents can be separated into their positive-, negative-, and zero-sequence components of current and many types of calculations simplified thereby. This method, which depends upon superposition, can be rigorously applied only when impedances and admittances are constant. It cannot be applied to the determination of proximity effect during unbalanced system operation, because proximity effect depends upon the relative magnitudes and phase relations between the *actual currents* in the conductors. An illustration of this is a solidly grounded three-phase system in which a line-to-ground fault occurs on an unloaded circuit where the three conductors are very close to each other. The current in the faulted phase may be very high while the currents in the two unfaulted phases are zero; and if the return path for current in the faulted phase is at a distance from that phase, there will be no appreciable proximity effect. If the proximity-effect resistance ratios for currents of each sequence were calculated and applied, an appreciable proximity effect in this circuit would be obtained, whereas there is actually no proximity effect.

With a line-to-line fault on a three-phase circuit, where the current in one phase returns in the other and there is no current in the third phase, proximity effect will be the same as for a single-phase circuit consisting of the two faulted phases.

In more complicated cases of unbalanced operation, the effect of proximity is not so readily analyzed as in the two examples given above. In any case where the phase currents are zero-sequence currents only, their proximity effect will be *zero-sequence proximity effect*, which is to be distinguished from positive-sequence proximity effect. Proximity effect under normal balanced operation is *positive-sequence proximity effect.* The curves given in this chapter apply to positive-sequence proximity effect. No reference to the calculation of zero-sequence proximity effect has been found in the literature; this is not surprising, as the condition of currents, equal and in phase in the three conductors, is rare except in staged tests.

**Positive-Sequence Self-Impedance.** Consider the positive-sequence self-impedance $Z_{11}$ in a three-phase or a single-phase circuit given by [59] of Chapter I in ohms per 1000 feet, in which constant frequency and uniform current distribution in the conductors are assumed:

$$\begin{aligned} Z_{11} &= r + jx = r + j2\pi fL = r + j2\pi f(L_e + L_i) \\ &= r + j\left(0.0529\,\frac{f}{60}\log_{10}\frac{2s}{d} + x_i\right) = r + j(x_e + x_i) \qquad [18] \end{aligned}$$

where $r$ and $x_i$ $(=2\pi fL_i)$ are resistance and internal reactance of one conductor; $L = L_e + L_i$ is total inductance to neutral; $L_e$ is external inductance to neutral; and $L_i$ is internal inductance. External reactance $x_e = 2\pi fL_e$ is given by the first term in the parentheses in [18].

Under the assumption of uniform current distribution in the conductors, the effective a-c resistance $r$ in [18] is the d-c resistance in ohms per 1000 feet of one conductor at the given temperature; the internal reactances of hollow non-magnetic cylindrical conductors at 60 cps are given in Fig. 1 in ohms per 1000 feet; the internal reactances[4] at 60 cps

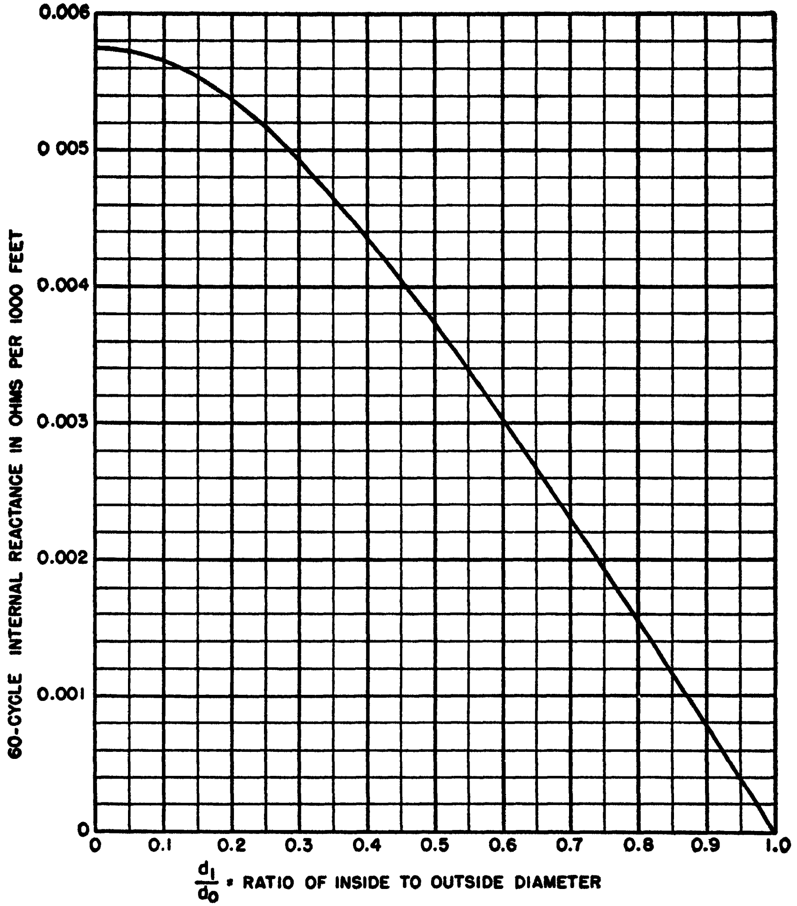

Fig. 1. Sixty-cycle internal reactance of hollow non-magnetic conductors (uniform current distribution).

of non-magnetic conductors of concentric stranding are given in Table II. The internal reactances of Fig. 1 and Table II vary directly with frequency.

With equal and opposite currents in the two conductors of a single-phase circuit, and the conductors far enough apart for proximity effect to be negligible, skin effect tends to increase $r$ and decrease $L_i$ (and

### TABLE II

### INTERNAL REACTANCE AT 60 CYCLES PER SECOND OF NON-MAGNETIC CONDUCTORS OF CONCENTRIC STRANDING, ASSUMING UNIFORM CURRENT DISTRIBUTION

| Number of Strands | $x_i$, Ohms | |
|---|---|---|
| | Per Mile | Per 1000 Feet |
| 7 | 0.0390 | 0.00740 |
| 19 | 0.0337 | 0.00638 |
| 37 | 0.0321 | 0.00608 |
| 61 | 0.0314 | 0.00595 |
| Solid | 0.0303 | 0.00575 |

$x_i$) in [18] from their values based on uniform current distribution. As the conductors are moved closer together, $r$ increases and the total inductance to neutral, $L = L_e + L_i$, decreases from the values of $r$ and $L$ which existed with negligible proximity effect. The influence of both skin effect and proximity effect upon resistance is much greater than upon inductance. Skin effect and proximity effect *resistance ratios* have received more attention in the literature than the corresponding inductance ratios.

In a single-phase or three-phase circuit of identical parallel conductors, operating under balanced conditions, let

$R_{dc}$ = d-c resistance of the conductors at the specified temperature

$R'$ = effective a-c resistance, skin effect included

$R''$ = effective a-c resistance, skin effect and proximity effect both included = $r$

$L_i$ = internal inductance of the conductors, assuming uniform current distribution

$L_e$ = external inductance to neutral, assuming uniform current distribution

$L_i'$ = internal inductance, skin effect included

$L'$ = inductance to neutral, skin effect on internal inductance included

$L''$ = total inductance to neutral, skin effect and proximity effect both included

The *effective a-c resistance* $r$ is

$$r = R_{dc} \times \frac{R'}{R_{dc}} \times \frac{R''}{R'} \qquad [19]$$

where $R'/R_{dc}$ = *skin-effect resistance ratio*, $R''/R'$ = *proximity-effect resistance ratio.*

The internal inductance $L_i'$ with skin effect included is

$$L_i' = L_i \times \frac{L_i'}{L_i} \qquad [20]$$

where $L_i'/L_i$ = *skin-effect internal-inductance ratio.*

The inductance to neutral $L'$ with skin effect included is

$$L' = L_e + L_i' \qquad [21]$$

The inductance to neutral $L''$ with both skin effect and proximity effect included is

$$L'' = L' \times \frac{L''}{L'} = (L_e + L_i') \times \frac{L''}{L'} \qquad [22]$$

where $L''/L'$ = *proximity-effect inductance ratio.*

When skin effect and proximity effect are both present, these effects upon resistance and inductance can be determined separately as ratios; then, these ratios can be applied as indicated in the above equations to determine effective a-c resistance and inductance to neutral. When proximity effect is negligible, both $R''/R'$ and $L''/L'$ are unity. For convenience in calculation, skin effect and proximity effect are treated separately although they are the same phenomenon — that associated with non-uniform current distribution in conductors. Separating them as above and then combining their effects is not a case of superposition, and no approximation is involved. For example, $R''$ as defined is the effective a-c resistance corresponding to the actual current distribution in the conductors and the actual flux distribution both inside and outside the conductors. $R'$ is the effective a-c resistance when proximity effect is negligible but skin effect has been taken into account. If $R'$ has been determined,

$$R'' = R' \times \frac{R''}{R'} = r = \text{effective a-c resistance}$$

**Homogeneous Cylindrical Conductors.** Formulas have been developed and charts prepared for the following skin-effect and proximity-effect ratios:

I. $R'/R_{dc}$ and $L_i'/L_i$ = skin-effect resistance and internal-inductance ratios for solid conductors of constant permeability (Fig. 2).

II. $R'/R_{dc}$ = skin-effect resistance ratios for tubular (and solid) non-magnetic conductors (Fig. 3).

III. $R''/R'$ = positive-sequence proximity-effect resistance ratios for single-phase and three-phase circuits of identical non-magnetic solid and tubular conductors [Figs. 4(*a*) and 4(*b*)].

IV. $L''/L'$ = positive-sequence proximity-effect inductance ratios for single-phase circuits of identical non-magnetic solid conductors (Fig. 5).

## Skin Effect

**Skin-Effect Resistance and Internal-Inductance Ratios in Solid Homogeneous Cylindrical Conductors of Constant Permeability.** The ratios $R'/R_{dc}$ and $L_i'/L_i$ at any temperature are functions of $\mathbf{x}$, where

$$\mathbf{x} = \pi d\sqrt{\frac{2f\mu}{\rho}} \qquad [23]$$

$$= \sqrt{\frac{8\pi f\mu\left(\frac{\pi}{4}d^2\right)}{\rho}} = \sqrt{\frac{8\pi f\mu A}{\rho}} \qquad [23a]$$

where $d$ = diameter of conductor in centimeters
$f$ = frequency in cycles per second
$\mu$ = permeability, assumed constant and uniform
$\rho$ = resistivity of material of conductor in abohms (cm cube) at conductor temperature
$A$ = cross section of conductor in square centimeters = $\pi d^2/4$.

E. B. Rosa and F. W. Grover[5] give a table of the functions of $\mathbf{x}$ as defined in [23] in small steps from which $R'/R_{dc}$ and $L_i'/L_i$ for any value of $\mathbf{x}$ between 0 and ∞ can be obtained. The curves of Fig. 2, plotted directly from this table, give the skin-effect resistance ratio $R'/R_{dc}$ and the internal-inductance ratio $L_i'/L_i$ versus $\mathbf{x}$. For convenience in using Fig. 2, $\mathbf{x}$ as defined by those authors has been expressed in terms of $\sqrt{f/R_{dc}}$ (a procedure followed in reference 6) where $R_{dc}$ is the d-c resistance of the conductor at the given temperature in ohms per 1000 feet.

In [7], if $\rho_v$ is in abohms (cm cube) and $\ell$ and $A$ are in centimeters and square centimeters, respectively, $R$ will be in abohms. Let $R_{dc}$ = *d-c resistance of the conductor in ohms per* 1000 *feet* at the given temperature. From [7],

$$\frac{\rho}{A} = \frac{R}{l} = \frac{R_{dc} \times 10^9}{30.48 \times 10^3} = R_{dc}\frac{10^6}{30.48} \text{ abohms per cm} \qquad [24]$$

Substitution of [24] in [23*a*] gives

$$\mathbf{x} = 0.02768\sqrt{\frac{f\mu}{R_{dc}}} \qquad [25]$$

where $R_{dc}$ is d-c resistance of the conductor in ohms per 1000 feet at conductor temperature and permeability $\mu$ is constant and uniform.

In Fig. 2, the ordinate is $R_c/R_{dc}$ or $L_i'/L_i$; the abscissa is $\mathbf{x}$, defined in [23] and also in [25]. The calculated value of $\mathbf{x}$ will be the same regardless of which equation is used. The quantity $\mathbf{x}$ as defined by Rosa and Grover has been retained in Fig. 2 for convenience when referring to their paper.[5] By calculating $\mathbf{x}$ from [25] instead of [23], the necessity of considering the change of resistivity and dimensions with temperature is avoided, as $R_{dc}$ can be read from wire tables or calculated at any given conductor temperature.

Figure 2 is applicable to *solid homogeneous cylindrical conductors* of *constant permeability* at any temperature and frequency within the range of the curves. $R'/R_{dc}$ is given in Fig. 2 for values of $\mathbf{x}$ between 0 and 6; $L_i'/L_i$ is given for values of $\mathbf{x}$ between 0 and 10. $R'/R_{dc}$ for values of $\mathbf{x}$ above 6 and $L_i'/L_i$ for values of $\mathbf{x}$ above 7 are given by the following approximate equations with less than 1% error, the error decreasing as $\mathbf{x}$ is increased:

$$\frac{R'}{R_{dc}} = \frac{\mathbf{x}}{2\sqrt{2}} + \frac{1}{4} \qquad [26]$$

$$\frac{L_i'}{L_i} = \frac{2\sqrt{2}}{\mathbf{x}} \qquad [27]$$

**Solid Magnetic Conductors.** The permeability of conductors of a given magnetic material varies with current, frequency, and manufacturing processes. A reliable value of $\mu$ for substitution in [23] or [25] is therefore seldom available. Test values of effective a-c resistance and internal reactance should be used when available (Volume I, pages 517–519).

**Solid Non-Magnetic Conductors.** For conductors of copper and other non-magnetic materials, $\mu = 1$ and [25] becomes

$$\mathbf{x} = 0.02768\sqrt{\frac{f}{R_{dc}}} \qquad [28]$$

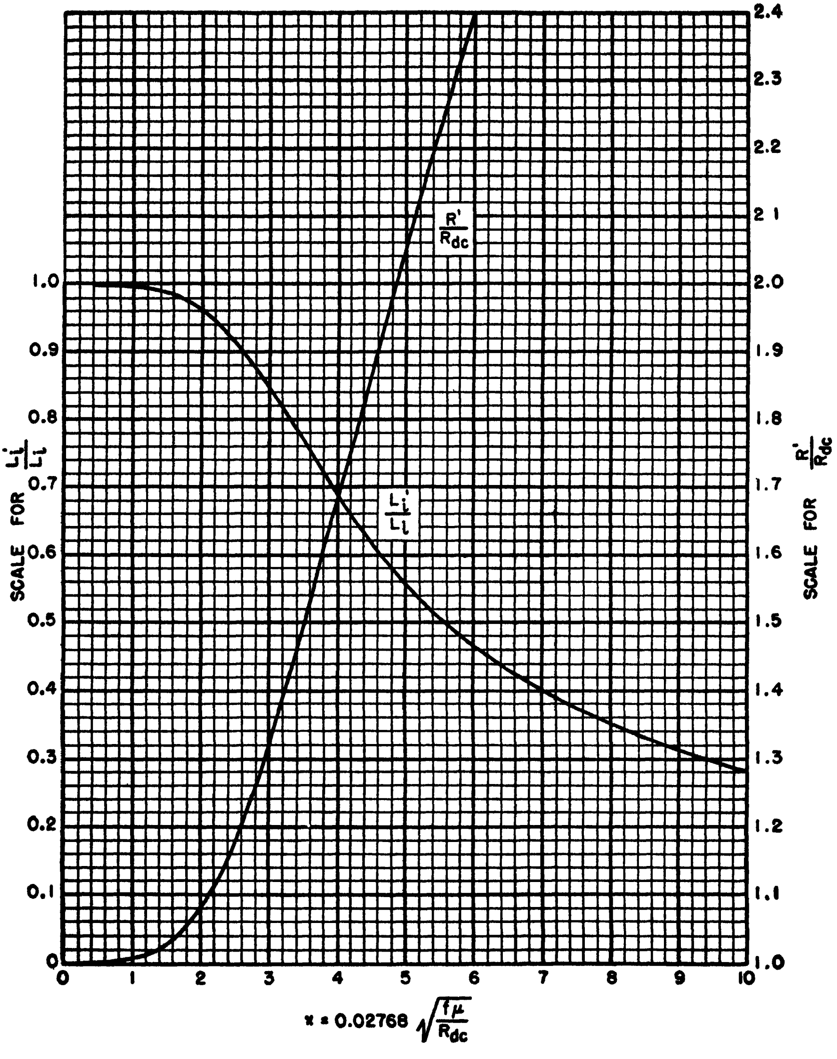

FIG. 2. Skin-effect resistance and internal-inductance ratios for solid round conductors of constant permeability. (E. B. Rosa and F. W. Grover, reference 5.)

**Example 2.** Find the skin-effect resistance and internal-inductance ratios of a solid 4/0 (212,000 cir mils) conductor of standard annealed copper at 25°C and 400 cps. Find its internal impedance $z_i = r + jx_i$ when proximity effect is negligible.

*Solution.* From [17] or Appendix A, $R_{dc} = 0.0500$ ohm per 1000 feet. From Table II, $x_i = (f/60)0.00575 = 0.0383$ ohm per 1000 feet. From [28], $x = 0.0277\sqrt{400/0.05} = 2.48$. From Fig. 2, $R'/R_{dc} = 1.17$ and $L_i'/L_i = 0.92$. The internal impedance $z_i$ in ohms per 1000 feet is

$$z_i = 0.0500(1.17) + j0.0383(0.92) = 0.0585 + j0.0352$$

**Skin-Effect Resistance Ratios in Tubular (and Solid) Non-Magnetic Cylindrical Conductors.** Figure 3 gives skin-effect resistance ratios $R'/R_{dc}$ in tubular conductors (with solid conductors a special case) versus $\sqrt{f/R_{dc}}$, where $R_{dc}$ is the d-c resistance of the conductor in ohms per 1000 feet at the given temperature. The parameter is $t/d$, the ratio of the thickness of the tube to its outside diameter. For a solid conductor, $t/d = 0.5$. Figure 3 is reproduced from Fig. 1 of

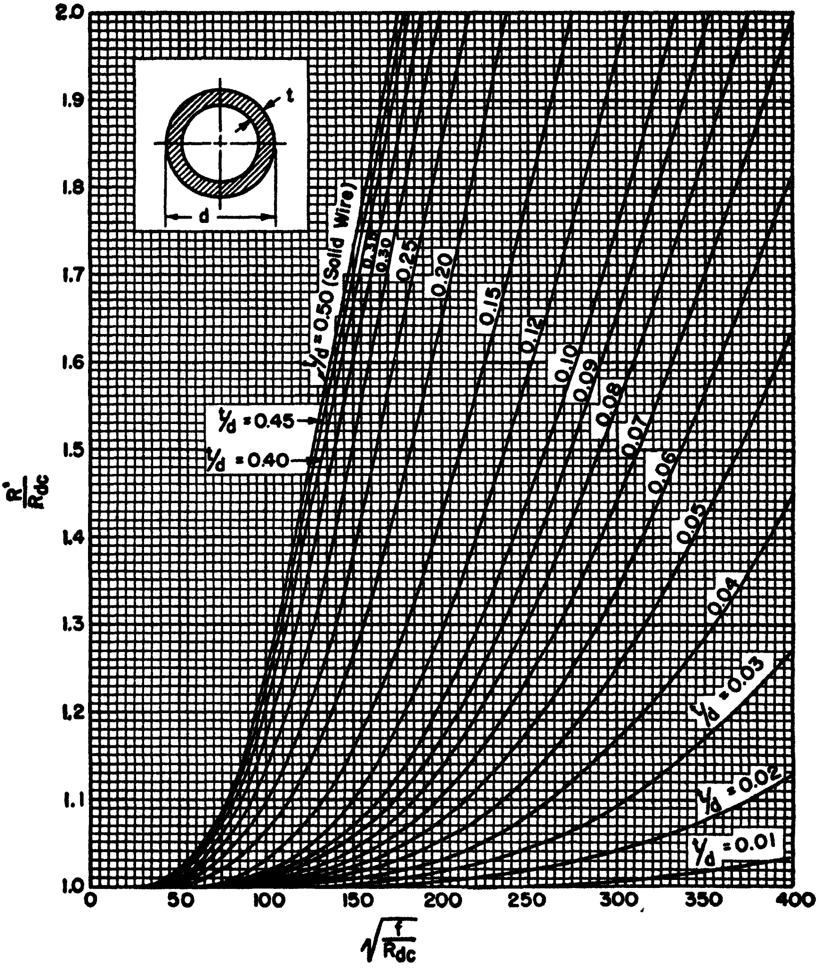

FIG. 3. Skin-effect resistance ratios in tubular and solid non-magnetic round conductors. (H. B. Dwight, reference 6.)

reference 6. Note that the abscissa is $\sqrt{f/R_{dc}}$, *not* $\mathbf{x} = 0.02768\sqrt{f/R_{dc}}$ as in Fig. 2 (if $\mu = 1$), Fig. 4, and Fig. 5.

From Fig. 3, the skin-effect resistance ratio in a tubular conductor is less than in a solid conductor of the same cross section of material. Tubular conductors also have the advantage of a larger diameter than the solid conductor of the same resistance and therefore a decreased voltage gradient at the surface. Hollow copper conductors and steel-reinforced aluminum cables are used on high-voltage overhead transmission lines to secure a diameter large enough to avoid corona without an unnecessary increase in conducting material.

The *internal reactance of a tubular conductor*, uniform current distribution being assumed, is less than that of a solid conductor. (See Fig. 1.) Any *change* in the internal reactance of a non-magnetic tubular conductor resulting from skin effect, being a change in a normally small part of the total reactance to neutral, may be neglected without appreciable error in power conductors at frequencies of 60 cps or less.

**Example 3.** The outside diameter of an overhead hard-drawn copper tubular conductor of 512,000 cir mils is 1.40 inches. What is its effective a-c resistance per 1000 feet at a frequency of 60 cps and a temperature of 75°C? There is no proximity effect.

*Solution.* Assuming a conductivity of 97.3% for hard-drawn copper, at 20°C the resistivity in ohms (mil, foot) is 10.371/0.973 = 10.66; from [1], the constant-mass temperature coefficient of resistance at 20°C is 0.00382. In ohms per 1000 feet at 75°C,

$$R_{dc} = \frac{10.66 \times 10^3}{512{,}000}[1 + 0.00382(75 - 20)] = 0.0252$$

$$\sqrt{\frac{f}{R_{dc}}} = \sqrt{\frac{60}{0.0252}} = 49$$

The outside diameter $d_o = 1.40$ inches. The inside diameter $d_i$ is

$$d_i = 10^{-3}\sqrt{(1.4)^2 \times 10^6 - 512{,}000} = 1.2033 \text{ inches}$$

$$t = 0.0983 \text{ in.}; \quad \frac{t}{d} = 0.0702$$

From Fig. 3, with $\sqrt{f/R_{dc}} = 49$ and $t/d = 0.070$, $R'/R_{dc} = 1.0$. Skin effect is negligible for this conductor at 60 cps. Therefore, at 75°C,

$$R' = R_{dc} = 0.0252 \text{ ohm per 1000 feet}$$

**Range of $\sqrt{f/R_{dc}}$ and x in 60-Cycle Circuits.** Table III gives the range of $\sqrt{f/R_{dc}}$ and $\mathbf{x}$ at a frequency of 60 cps for solid and tubular conductors of standard annealed copper at 25°C. In this table, $R_{dc}$

TABLE III

VALUES OF $\sqrt{f/R_{dc}}$ AND $x = 0.02768\sqrt{f/R_{dc}}$ AT 60 CPS AND 25°C FOR SOLID AND TUBULAR ROUND CONDUCTORS OF STANDARD ANNEALED COPPER

| Cir Mils | $R_{dc}$, Ohms per 1000 Feet | $\sqrt{f/R_{dc}}$ | $x = 0.0277\sqrt{f/R_{dc}}$ |
|---|---|---|---|
| 4/0 | 0.0500 | 34.6 | 0.96 |
| 250,000 | 0.04230 | 37.7 | 1.04 |
| 500,000 | 0.02115 | 53.4 | 1.48 |
| 750,000 | 0.01410 | 65.2 | 1.81 |
| 1,000,000 | 0.01057 | 75.3 | 2.09 |
| 2,000,000 | 0.00529 | 106.5 | 2.95 |
| 3,000,000 | 0.00352 | 130 | 3.62 |
| 4,000,000 | 0.00264 | 151 | 4.17 |
| 5,000,000 | 0.00211 | 168 | 4.66 |

does not include the increase in resistance due to the greater length of the strands than of the finished stranded conductor. At temperatures above 25°C, $x$ will be smaller than at 25°C. For conductors up to 750,000 cm, $x$ in Table III is less than 2; for larger conductors up to 5,000,000 cir mils, $x$ is less than 5.

## Spirality Effect

The curves in Fig. 2 are plotted from formulas based on *solid conductors*, those in Fig. 3 from formulas based on tubular conductors (non-stranded). Before these curves can be applied to stranded conductors, it is necessary to consider the effect of spirality on skin-effect ratios. It has been shown that the skin-effect resistance and internal-inductance ratios of a conductor, subdivided into parallel *unspiraled* strands which touch each other, are the same as for a solid conductor of the same material and cross section.[7] Any change in these ratios therefore results from spiraling and not from stranding. Spirality effect may either increase or decrease the regular skin-effect ratios, depending upon the cross section of the conductor, the number of strands, the method of spiraling, and the frequency.

*Spirality-effect resistance ratio* is the factor by which the skin-effect resistance ratio of a solid conductor is multiplied to obtain the skin-effect resistance ratio of a stranded conductor of the same cross section of material. Tests[7] on a 4/0 copper conductor of seven strands, six of these being spiraled around the center one (concentric stranding), pitch of spiral approximately 14.5 cm, at frequencies between 60 cps and 5040 cps give a spirality-effect resistance ratio of approximately unity at 60 cps and also at approximately 1250 cps. Between these two frequencies, the ratio is slightly less than unity (between 0.987 and 1.00). This ratio gradually increases above 1250 cps, reaching approximately 1.06 at 5040 cps. Subsequent tests[8] made at frequencies between 60 cps and 5000 cps on solid, 7-strand, 19-strand, and 37-strand 4/0 copper conductors of concentric stranding show (by means of curves, Fig. 1, reference 8) that the 19-strand and 37-strand conductors have lower skin-effect resistance ratios than either the 7-strand or the solid conductor, the 37-strand conductor having the lowest. At 60 cps, the spirality-effect resistance ratios for the 19-strand and the 37-strand conductors are slightly less than unity; at higher frequencies up to 5000 cps, this ratio is appreciably less than unity. From the curves, it may be seen that the 19-strand conductor will have unity spirality-effect ratio at some frequency above 5000 cps; a unity ratio for the 37-strand conductor, if it occurs, will be at a very much higher frequency. From these tests, the conclusion can be drawn that increasing the number of strands in 4/0 copper conductors of *concentric stranding* (provided this number is greater than seven) *decreases* the skin-effect resistance ratio for frequencies at least up to 5000 cps.

These data[7,8] on 4/0 copper conductors may be applied[9] to other conductors of the same type of stranding and number of strands, at the same values of $\mathbf{x}$ or $\sqrt{f/R_{dc}}$. For a solid 4/0 copper conductor of standard annealed copper at 25°C and 5000 cps, $\sqrt{f/R_{dc}} = \sqrt{5000/0.05} = 316$ and $\mathbf{x} = 0.02768\sqrt{f/R_{dc}} = 8.75$. On the basis of the test data discussed above, up to these values of $\sqrt{f/R_{dc}}$ and $\mathbf{x}$, the spirality-effect resistance ratio *will not be greater than unity* for conductors of *concentric stranding* having more than 7 strands; for conductors of 7 strands, the spirality-effect resistance ratio will not exceed unity for values of $\sqrt{f/R_{dc}}$ and $\mathbf{x}$ up to 158 and 4.4, respectively. Within these limits of $\sqrt{f/R_{dc}}$ and $\mathbf{x}$, and for conductors of concentric stranding, the skin-effect resistance ratio with spirality effect included *will not be greater* than that read from Fig. 2 for a solid conductor of the same circular mils as the given stranded conductor.

For frequencies of 60 cps or less, the spirality-effect resistance ratio may be taken as approximately unity in a conductor of concentric stranding. The skin-effect resistance ratio is then read directly from Fig. 2 or Fig. 3, where $R_{dc}$ is the d-c resistance of the given stranded conductor or of the solid conductor of the same circular mils of cross section. If $R_{dc}$ of the stranded conductor, which is 2% higher than

$R_{dc}$ of the solid conductor of the same cross section of material, is used to calculate $\sqrt{f/R_{dc}}$ and **x**, $\sqrt{f/R_{dc}}$ and **x** will be approximately 1% lower, and the skin-effect resistance ratio $R'/R_{dc}$ read from Fig. 2 or Fig. 3 will be smaller than that read with **x** and $\sqrt{f/R_{dc}}$ calculated from $R_{dc}$ of the solid conductor of the same circular mils. This is a minor difference in 60-cycle systems, mentioned here to insure that the same value of $R_{dc}$ is used when values read from curves based on equations given by different authors are compared. See Example 4.

For frequencies higher than 60 cps and for conductors of stranding and spiraling different from those of *concentric stranding* described in this chapter, additional test values of spirality-effect resistance ratios are required before Figs. 2 and 3 can be applied otherwise than approximately.

It has been pointed out,[7] that spirality effect and also skin effect can be substantially eliminated if the separate strands are insulated from each other and completely transposed so that, in its turn, each strand occupies each position in the cross section.

**Stranded Annular Conductors.** The d-c resistances of stranded annular conductors with rope centers are from 3% to 5% higher than those of tubular conductors of the same circular mils. (See Table IV, Appendix A.) On the assumption that the spirality-effect resistance ratio for these conductors at 60 cps is never greater than unity, $R_{dc}$ of the given annular conductor can be used to calculate $\sqrt{f/R_{dc}}$ in Fig. 3. Actually, the difference in the skin-effect resistance ratios read from Fig. 3 by using $R_{dc}$ of the stranded annular conductor and $R_{dc}$ of the tubular conductor (non-stranded) of the same circular mils is very slight. See Example 5.

**Example 4.** Find $R'/R_{dc}$ and $L_i'/L_i$ for an annealed copper conductor of standard concentric stranding, 1,000,000 cir mils, at 60 cps and 70°C, assuming that spirality effect is negligible.

*Solution.* For a *solid conductor* of 1,000,000 cir mils, $R_{dc}$ per 1000 feet is

$$R_{dc} = \frac{10.371 \times 10^3}{1 \times 10^6}[1 + 0.00393(70 - 20)] = 0.0124$$

From [28], $\mathbf{x} = 0.0277\sqrt{60/0.0124} = 1.93$. From Fig. 2, with $\mathbf{x} = 1.93$, $R'/R_{dc} = 1.07$; $L_i'/L_i = 0.97$. From Fig. 3, with $\sqrt{f/R_{dc}} = \sqrt{60/0.0124} = 69.6$ and $t/d = 0.5$, $R'/R_{dc} = 1.07$.

If $R_{dc}$ of the stranded conductor is used to calculate $\sqrt{f/R_{dc}}$, $R'/R_{dc}$ will be approximately 1.065.

**Example 5.** Required: the effective 60-cycle a-c resistance at 70°C, when proximity effect is negligible, of an annular stranded annealed copper conductor of 1,000,000 cir mils (nominal), 1,023,766 cir mils (actual), inside diameter 0.563 inch, outside diameter 1.346 inches. The d-c resistance is increased 3% to allow for stranding and spiraling.

*Solution.* Multiplying [17] by 1.03 to allow for greater length of the strands than of the finished conductor,

$$R_{dc} = 1.03 \frac{10.371 \times 10^3}{1{,}023{,}766} [1 + 0.00393(70 - 20)]$$

$$= 1.03 \times 0.01212 = 0.0125 \text{ ohm per 1000 feet for stranded annular conductor,}$$

$$R_{dc} = 0.01212 \text{ ohm per feet for the solid conductor of same circular mils}$$

With $R_{dc} = 0.01212$, $\sqrt{f/R_{dc}} = 70.5$. With $R_{dc} = 0.0125$, $\sqrt{f/R_{dc}} = 69.5$.

$$\frac{t}{d} = \frac{1.346 - 0.563}{2(1.346)} = 0.29$$

From Fig. 3 with $t/d = 0.29$ and $\sqrt{f/R_{dc}} = 70.5$ or $69.5$, $R'/R_{dc} = 1.035$; $R' = 1.035 \times 0.0125 = 0.0129$ ohm per 1000 feet.

## Proximity Effect

The phenomenon of non-uniform current distribution in conductors has been divided for convenience into skin effect and proximity effect. Skin effect relates to the phenomenon of non-uniform current distribution which is symmetrical about the axis of a round conductor. In a circuit in which the conductors are far enough apart for the current in any conductor of the circuit to have no appreciable effect on the current distribution in the other conductors, proximity effect is negligible and skin effect only need be considered. In such a circuit, the conductors are referred to as isolated. In overhead transmission circuits of conventional design, proximity effect is negligible. In the tables of Appendix B, Volume I, skin effect on resistance and internal inductance has been taken into account.

Proximity effect combined with skin effect is a more complicated phenomenon than skin effect alone. The current distribution in the conductors and the resulting flux distribution both inside and outside the conductors when both effects are present is unsymmetrical about the axes of the conductors. As defined under [19], the proximity-effect resistance ratio $R''/R'$ is the ratio of the effective a-c resistance with both proximity effect and skin effect taken into account to the effective a-c resistance when the conductors are far enough apart for proximity effect to be negligible. Equations for the proximity-effect resistance ratios in tubular conductors involve all the independent variables encountered in determining skin-effect ratios; and, in addition, the spacing between conductors introduces an extra variable.

**Proximity-Effect Resistance Ratios for Single-Phase and Three-Phase Circuits of Identical Non-Magnetic Solid and Tubular Homogeneous Round Conductors.** In a comprehensive treatment of proximity-effect resistance ratios in single-phase and three-phase circuits under balanced operation, A. H. M. Arnold[10] has given equations and tabulated functions of $\mathbf{x}$ (defined below) for determining the proximity-effect resistance ratio $R''/R'$ in a single-phase circuit of two identical tubular conductors, with solid conductors a special case. He also gives factors to be applied to $R''/R'$ in the single-phase circuit to determine $R''/R'$ in a three-phase circuit with symmetrical triangular spacing, and the average $R''/R'$ in a three-phase circuit with flat spacing in which the center conductor is equidistant from the other two. The following assumptions are made: (*a*) the conductors are non-magnetic and long enough for end effects to be negligible; (*b*) the effects of capacitance and leakance can be neglected; (*c*) the magnetic field is perpendicular to the axes of the conductors and does not vary along the axes.

The paper shows a highly satisfactory agreement between test and calculated values of effective a-c resistance over a wide range of the independent variables involved. To reduce the time and effort required in calculations, charts are given in this chapter from which proximity-effect resistance ratios can be quickly determined. These charts are based on the equations and tables, given in the paper,[10] which are discussed briefly below.

**Single-Phase Circuit of Two Identical Conductors.** The proximity-effect resistance ratio $R''/R'$ in tubular conductors, with solid conductor a special case, depends upon three variables: $t/d$, $d/s$, and $\mathbf{x}$, where

$t/d$ = ratio of thickness $t$ of tube to its outside diameter $d$ ($t/d = 0.5$ for a solid conductor)

$d/s$ = ratio of outside diameter $d$ of conductor to distance $s$ between axes of conductors

The independent variable $\mathbf{x}$, defined in the paper in terms of $t$, $d$, and $\rho$ in absolute units, is

$$\mathbf{x} = 2\pi\sqrt{\frac{2ft(d-t)}{\rho}} \qquad [29]$$

where $\rho$ is the resistivity of the material in abohms (cm cube), and $d$ and $t$ are in centimeters, all at conductor temperature, and $f$ is frequency in cycles per second.

Equation [29] may be rewritten:

$$\mathbf{x} = \sqrt{\frac{8\pi f}{\rho}\left(\frac{\pi}{4}d^2\right)\left[\frac{4t}{d}\left(1 - \frac{t}{d}\right)\right]} = \sqrt{\frac{8\pi fA}{\rho}} \qquad [30]$$

where the area of the cross section $A$ in square centimeters in terms of $t$ and $d$ in centimeters is given by [16] for a tubular conductor. If [30] and [23$a$] are compared when $\mu = 1$, they are seen to be the same. Replacing $\rho/A$ in abohms per centimeter by its value given in [24] in terms of $R_{dc}$,

$$\mathbf{x} = 0.02768\sqrt{\frac{f}{R_{dc}}} \qquad [31]$$

where $R_{dc}$ = d-c resistance of the conductor in ohms per 1000 feet at conductor temperature.

For those who wish to consult the reference paper, it is emphasized that $\mathbf{x}$ calculated by [31] will be exactly the same as $\mathbf{x}$ calculated by [29], the equation given by the author.[10] The use of [31] instead of [29], like the use of [28] instead of [23], eliminates the necessity of considering change of resistivity and dimensions with temperature for conductors of constant mass.

The proximity-effect resistance ratio $R''/R'$ for a single-phase circuit in which the current in one of two identical conductors returns in the other is

$$\frac{R''}{R'} = \frac{1}{\sqrt{1 - \left(\frac{d}{s}\right)^2 A + \frac{\left(\frac{d}{s}\right)^4 B}{1 - \left(\frac{d}{s}\right)^2 C}}} \qquad [32]$$

where $R''/R'$ is the ratio of the effective a-c resistance of either conductor to that of an equal isolated conductor (see [19]). $A$, $B$, and $C$ are functions of $\mathbf{x}$ and $t/d$ which can be determined from tables given in the paper.[10] When $d/s$ is 0.5 or less than 0.5, the following equation (here called approximate) is given:

$$\frac{R''}{R'}\text{(approx.)} = \frac{1}{\sqrt{1 - \left(\frac{d}{s}\right)^2 A}} \qquad [33]$$

**Three-Phase Circuits.** The proximity-effect resistance ratio for the three-phase circuit carrying positive-sequence currents can be obtained from $R''/R'$ of the single-phase circuit having the same spacing, $s$ between axes of conductors by applying a multiplier $F$.

With *equilateral triangular spacing* $s$ between axes of conductors, the multiplier $F_\Delta$ is

$$F_\Delta = 1 + \frac{1}{4}\left(\frac{d}{s}\right)^2 A - \frac{5}{24}\left(\frac{d}{s}\right)^4 A^2 - \frac{3}{8}\left(\frac{d}{s}\right)^{16} A^8 \qquad [34]$$

With *flat spacing* and the axis of the center conductor at a distance $s$ from the axis of each of the outside conductors, the average multiplier $F_{\text{ave}}$ for the three conductors is

$$F_{\text{ave}} = 1 + \frac{1}{4}\left(\frac{d}{s}\right)^2 A - \frac{1}{8}\left(\frac{d}{s}\right)^4 A^2 + \frac{5}{24}\left(\frac{d}{s}\right)^{16} A^8 \qquad [35]$$

The multipliers $F_c$ and $F_o$ for the center and outer conductors are, respectively,

$$F_c = 1 + \left(\frac{d}{s}\right)^2 A \qquad [36]$$

$$F_o = 1 - \frac{1}{8}\left(\frac{d}{s}\right)^2 A - \frac{3}{16}\left(\frac{d}{s}\right)^4 A^2 + \frac{5}{16}\left(\frac{d}{s}\right)^{16} A^8 \qquad [37]$$

The positive-sequence proximity-effect resistance ratio in the three phase circuit is

$$\frac{R''}{R'}\text{ (three-phase)} = F\left[\frac{R''}{R'}\text{ (single-phase)}\right] \qquad [38]$$

where $R''/R'$ (single-phase) is given by [32], and $F$ by [34]–[37].

## Proximity-Effect Resistance-Ratio Charts

Equation [32] does not lend itself readily to the construction of charts; but [33]–[38] contain only the variables $A$ and $d/s$ and are therefore easily plotted. Let the ratio of $R''/R'$, calculated from [32], to $R''/R'$ (approx.), calculated from [33], be $K_c$; then,

$$\frac{R''}{R'} = K_c\left[\frac{R''}{R'}\text{ (approx.)}\right] \qquad [39]$$

where $K_c$ may be regarded as a correcting factor to be applied to $R''/R'$ (approx.) calculated from [33] to obtain $R''/R'$ given by [32].

$K_c$ is a function of $\mathbf{x}$, $t/d$, and $d/s$. When $K_c$ is unity, $R''/R'$ (approx.) from [32] and $R''/R'$ from [33] are the same. When $d/s \leqq 0.5$, $K_c$ is unity for all values of $\mathbf{x}$ and $t/d$.

Table IV gives $K_c$ for $\mathbf{x}$ between 0 and 10 with various values of $s/d$ and $t/d$ from which $K_c$ can be read directly or determined by interpolation. For $d/s = 0.6$ and 0.7, $K_c$ at a given value of $\mathbf{x}$ is substantially the same for all values of $t/d$. For $d/s = 0.8$, $K_c$ is given for $t/d = 0.5$ (solid conductors) and $t/d = 0$ (tubes of zero thickness). For $d/s =$ 0.9, 0.95, and 1.00, $K_c$ is given for $t/d = 0.5$, 0.1, and 0, except that $K_c$ for $t/d = 0$ and $d/s = 1.0$ is not given for values of $\mathbf{x}$ above 3. From this table, it may be noted that, except for values of $\mathbf{x}$ less than 1.8, $K_c$ is less than unity; if $\mathbf{x} \leqq 1.8$, the departure of $K_c$ from unity is less than 1% for all values of $t/d$ and $d/s$. For $d/s = 0.6$, 0.7, and 0.8, all departures of $K_c$ from unity are small, the maximum departures being 0.6, 1.3, and 2.5%, respectively, and they occur between $\mathbf{x} = 2.5$ and 3.5; for higher and lower values of $\mathbf{x}$, the departure of $K_c$ from unity is less than the maximum departure.

**Single-Phase Circuit.** For a single-phase circuit of two identical solid or tubular conductors, $A$ for substitution in [33] can be read directly from Fig. 4(*a*). In Fig. 4(*a*), $\mathbf{x}$ defined in [29] or [31] is abscissa, $A$ (a function of $\mathbf{x}$ and $t/d$) is ordinate, and $t/d$ is parameter. For values of $\mathbf{x}$ between 0 and 10, $A$ corresponding to $t/d = 0.5$, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01, or 0 can be read directly. The insert in Fig. 4(*a*) gives values of $A$ for $\mathbf{x}$ between 10 and 50 with $t/d = 0.5$, 0.01, and 0; for $\mathbf{x}$ between 10 and 20, curves for $t/d = 0.2$ and 0.05 are included.

When $A$ and $d/s$ are known, $R''/R'$ (approx.) can be calculated from [33] or read from Part II of Fig. 4(*b*), as explained later. $R''/R'$ for the single-phase circuit is then determined from [39], after $K_c$ has been read from Table IV.

**Three-Phase Circuit.** For a three-phase circuit of equilateral triangular spacings, or of flat spacing with the axis of the center conductor at a distance $s$ from the axis of each of the outside conductors, $A$ and $R''/R'$ for a single-phase circuit with the same distance $s$ between axes of conductors are determined as described above. $F_\Delta$ or $F_{\text{ave}}$ in terms of $A(d/s)^2$ is calculated from [34] or [35], or read from Part IV of Fig. 4(*b*). $R''/R'$ (three-phase) is then calculated from [38].

**Quick Estimating Chart.** Figure 4(*b*) is a composite chart for quick estimating purposes. It consists of four charts which are labeled I, II, III, and IV.

In Part I, lower left-hand corner of Fig. 4(*b*), $\mathbf{x}$ is plotted versus $\sqrt{f}$ with copper conductor of indicated sizes as parameter. The conductor resistances used in this chart are those for solid or tubular (nonstranded) standard annealed copper conductors at 25°C. The $\sqrt{f}$

## TABLE IV

### Correcting Factor $K_c$ to be Applied to Proximity-Effect Resistance Ratio Calculated from Approximate Equation [33]

| x | $d/s$ = 0.6 | $d/s$ = 0.7 | $d/s$ = 0.8 | | $d/s$ = 0.9 | | | $d/s$ = 0.95 | | | $d/s$ = 1.00 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $t/d$ 0.5 to 0 | $t/d$ 0.5 to 0 | $t/d$ 0.5 | $t/d$ 0 | $t/d$ 0.5 | $t/d$ 0.1 | $t/d$ 0 | $t/d$ 0.5 | $t/d$ 0.1 | $t/d$ 0 | $t/d$ 0.5 | $t/d$ 0.1 | $t/d$ 0 |
| 0 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.5 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 1.0 | 1.000 | 1.000 | 1.000 | 1.001 | 1.000 | 1.000 | 1.001 | 1.000 | 1.000 | 1.001 | 1.001 | 1.001 | 1.002 |
| 1.2 | 1.000 | 1.000 | 1.000 | 1.001 | 1.000 | 1.000 | 1.002 | 1.001 | 1.001 | 1.002 | 1.001 | 1.002 | 1.003 |
| 1.4 | 1.000 | 1.000 | 1.000 | 1.001 | 1.000 | 1.001 | 1.002 | 1.000 | 1.001 | 1.002 | 1.000 | 1.002 | 1.003 |
| 1.6 | 1.000 | 1.000 | 0.999 | 1.001 | 0.999 | 1.000 | 1.001 | 0.999 | 1.000 | 1.001 | 0.998 | 1.000 | 1.002 |
| 1.8 | 0.999 | 0.998 | 0.997 | 0.999 | 0.995 | 0.996 | 0.999 | 0.994 | 0.995 | 0.998 | 0.993 | 0.995 | 0.997 |
| 2.0 | 0.998 | 0.997 | 0.994 | 0.995 | 0.990 | 0.990 | 0.990 | 0.986 | 0.986 | 0.984 | 0.983 | 0.982 | 0.976 |
| 2.2 | 0.997 | 0.994 | 0.989 | 0.990 | 0.981 | 0.981 | 0.979 | 0.976 | 0.974 | 0.966 | 0.968 | 0.964 | 0.950 |
| 2.4 | 0.996 | 0.992 | 0.984 | 0.985 | 0.972 | 0.972 | 0.966 | 0.965 | 0.961 | 0.950 | 0.951 | 0.944 | 0.919 |
| 2.6 | 0.995 | 0.990 | 0.980 | 0.981 | 0.964 | 0.964 | 0.955 | 0.950 | 0.946 | 0.928 | 0.934 | 0.921 | 0.884 |
| 2.8 | 0.994 | 0.988 | 0.977 | 0.978 | 0.957 | 0.955 | 0.945 | 0.940 | 0.934 | 0.912 | 0.919 | 0.901 | 0.849 |
| 3.0 | 0.994 | 0.987 | 0.975 | 0.976 | 0.952 | 0.950 | 0.938 | 0.934 | 0.925 | 0.896 | 0.909 | 0.883 | 0.812 |
| 3.2 | 0.994 | 0.987 | 0.975 | 0.976 | 0.950 | 0.948 | 0.933 | 0.931 | 0.918 | 0.882 | 0.902 | 0.869 | |
| 3.4 | 0.995 | 0.988 | 0.976 | 0.976 | 0.951 | 0.947 | 0.931 | 0.930 | 0.916 | 0.873 | 0.900 | 0.859 | |
| 3.6 | 0.996 | 0.989 | 0.978 | 0.977 | 0.952 | 0.948 | 0.931 | 0.932 | 0.916 | 0.865 | 0.901 | 0.852 | |
| 3.8 | 0.996 | 0.990 | 0.979 | 0.978 | 0.957 | 0.950 | 0.932 | 0.935 | 0.917 | 0.863 | 0.903 | 0.850 | |
| 4.0 | 0.996 | 0.991 | 0.981 | 0.979 | 0.959 | 0.953 | 0.934 | 0.939 | 0.918 | 0.862 | 0.907 | 0.849 | |
| 4.5 | 0.997 | 0.993 | 0.985 | 0.983 | 0.966 | 0.959 | 0.942 | 0.948 | 0.927 | 0.868 | 0.915 | 0.852 | |
| 5 | 0.998 | 0.995 | 0.988 | 0.987 | 0.972 | 0.966 | 0.953 | 0.953 | 0.934 | 0.881 | 0.917 | 0.856 | |
| 6 | 0.998 | 0.996 | 0.991 | 0.993 | 0.976 | 0.975 | 0.970 | 0.956 | 0.947 | 0.915 | 0.914 | 0.865 | |
| 7 | 0.999 | 0.997 | 0.993 | 0.996 | 0.979 | 0.981 | 0.980 | 0.958 | 0.958 | 0.942 | 0.912 | 0.874 | |
| 8 | 0.999 | 0.998 | 0.994 | 0.997 | 0.981 | 0.985 | 0.987 | 0.960 | 0.964 | 0.959 | 0.909 | 0.883 | |
| 10 | 0.999 | 0.998 | 0.995 | 0.998 | 0.983 | 0.991 | 0.992 | 0.965 | 0.976 | 0.973 | 0.904 | 0.903 | |

scale extends from 0 to 20, including frequencies between 0 and 400 cps. Frequencies of 25, 60, 100, and 400 cps are indicated. For an indicated conductor size, **x** varies as the square root of the frequency.

In Part II, upper left-hand corner of Fig. 4(*b*), *A* is plotted versus **x** with $t/d$ as parameter. Part II is the same as Fig. 4(*a*), but to a smaller scale.

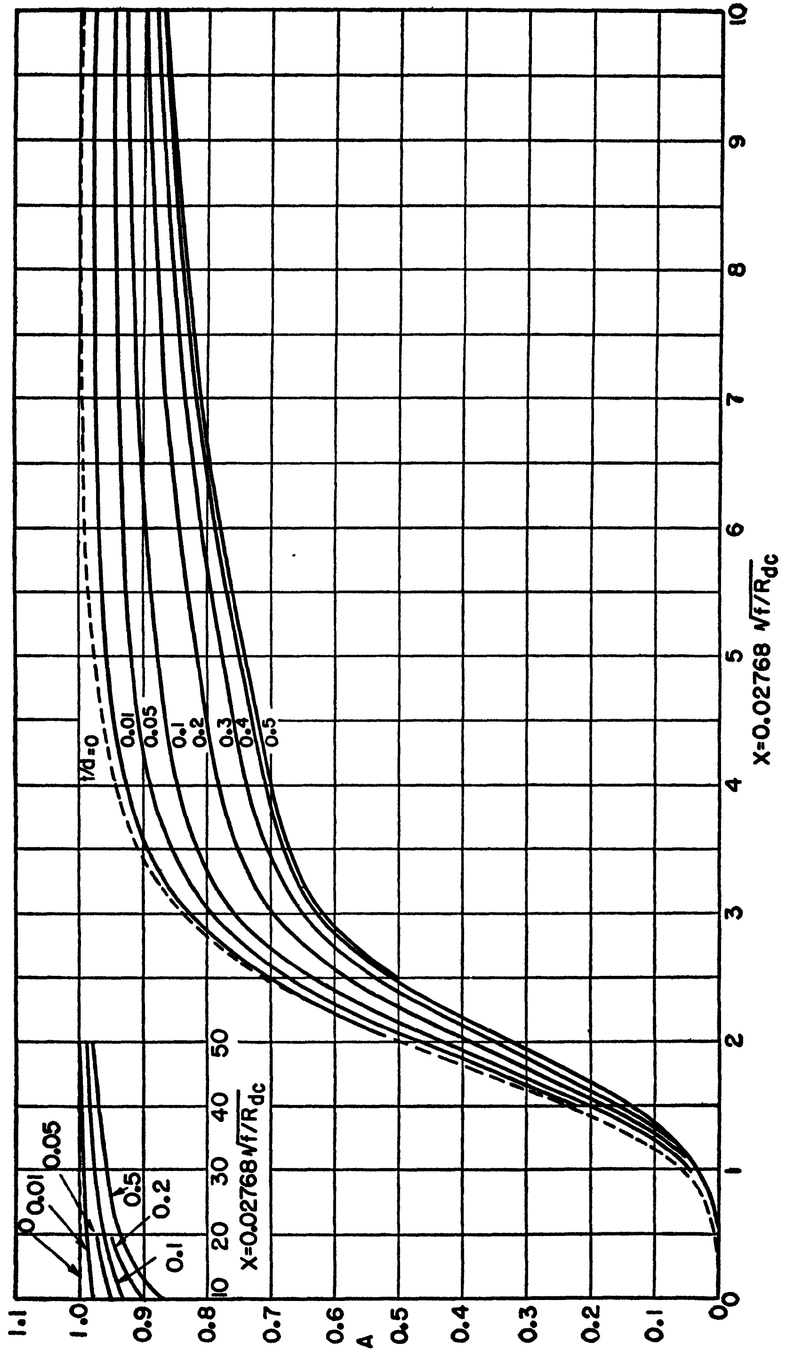

FIG. 4(*a*). Values of $A$ versus $\mathbf{x}$ with $t/d$ as parameter. (A. H. M. Arnold, reference 10.)

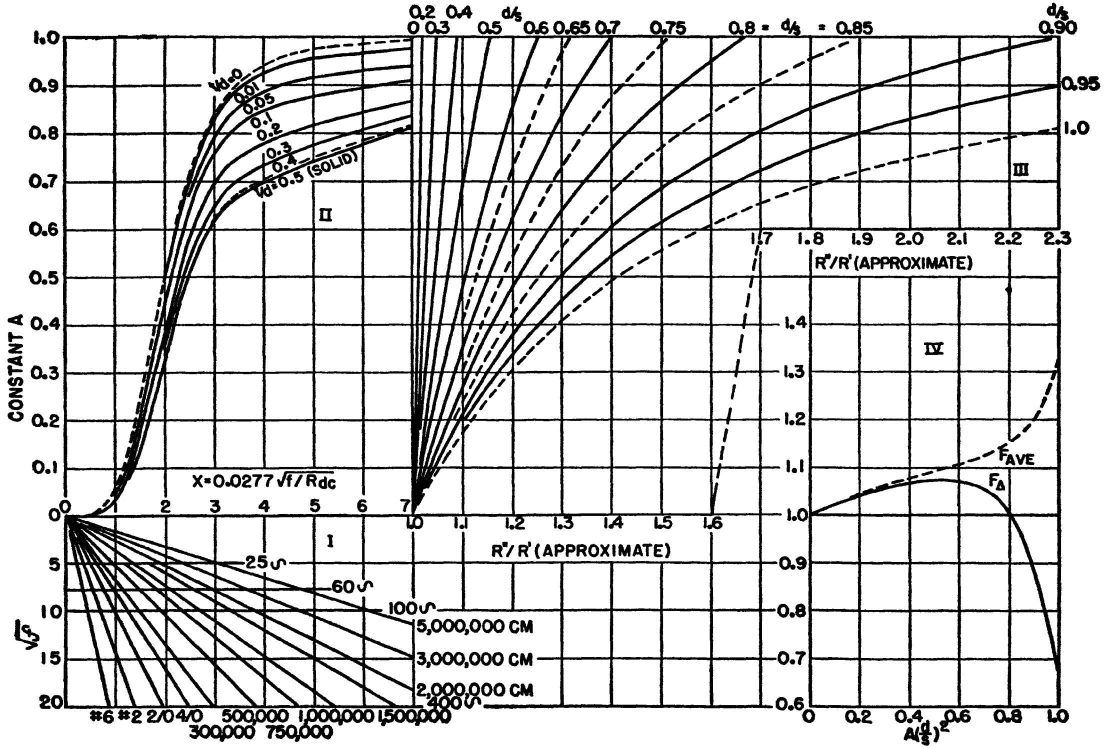

FIG. 4(*b*). Proximity-effect resistance-ratio chart — solid and tubular conductors. For single-phase circuit: $R''/R' = K_e$ (Table IV) $\times R''/R'$ (approx.). For three-phase circuit: $\Delta$ spacing, $R''/R' = R''/R$ (single-phase) $\times F_\Delta$; flat spacing, $R''/R' = R''/R'$ (single-phase) $\times F_{ave}$.

In Part III, upper right-hand corner of Fig. 4(*b*), $R''/R'$ (approx.) is plotted versus $A$ with $d/s$ as parameter.

In Part IV, lower right-hand corner of Fig. 4(*b*), $F_\Delta$ and $F_{\text{ave}}$ defined in [34] and [35], respectively, are plotted versus $A\,(d/s)^2$.

**Procedure for Use of Fig. 4(*b*) with Conductors of Annealed Copper at 25°C.** Calculate $\sqrt{f}$, $d/s$, and $t/d$ (if the conductor is tubular). For a solid conductor $t/d = 0.5$.

Locate the intersection of $\sqrt{f}$ and the conductor of the given size in Part I in the lower left-hand corner of Fig. 4(*b*). Read and record $\mathbf{x}$ if $d/s > 0.5$. Pass vertically upward to the calculated value of $t/d$ in Part II. Record $A$ if the circuit is a three-phase one. Pass horizontally to the calculated value of $d/s$ in Part III. Drop vertically downward and read $R''/R'$ (approx.) on the abscissa scale.

If $d/s > 0.5$, determine $K_c$ corresponding to $\mathbf{x}$, $t/d$, and $d/s$ from Table IV. Calculate $R''/R'$ (single-phase) from [39].

For the three-phase circuit, read $F_\Delta$ or $F_{\text{ave}}$ corresponding to $(d/s)^2A$ from Part IV of Fig. 4(*b*). Then calculate $R''/R'$ (three-phase) from [38].

*For conductors other than those of annealed copper at* 25°C the procedure is the same as given above except that $\mathbf{x}$ is calculated from [31], $R_{dc}$ of the given conductor at the given temperature being used, and Fig. 4(*b*) is entered at $\mathbf{x}$ in Part II. Part I is not used.

For annealed copper conductors of temperature higher than 25°C, $\mathbf{x}$ calculated from [31] will be lower than $\mathbf{x}$ given by Part I of Fig. 4(*b*).

**Stranded Conductors.** Although [32]–[38] are developed for solid and tubular (non-stranded) conductors, test values in the paper[10] indicate that they can be applied to stranded conductors. In this case, the outside diameter $d$ is the actual outside diameter of the stranded conductor, and the thickness $t$ is one-half the difference between outside and inside diameters. Within the degree of precision obtainable from the curves, it is unimportant whether $R_{dc}$ of the stranded conductor or $R_{dc}$ of the solid or tubular (non-stranded) conductor of the same circular mils is used in calculating $\mathbf{x}$. In Part I of Fig. 4(*b*), the conductors indicated are solid or tubular (non-stranded). As previously pointed out, $\mathbf{x}$ calculated from $R_{dc}$ of the stranded conductor will be approximately 1% lower than $\mathbf{x}$ calculated from $R_{dc}$ of the solid conductor of the same circular mils.

**Example 6.** In a single-phase circuit of solid 4/0 copper conductor of standard annealed copper, operated at 400 cps and 25°C, the distance between conductor centers is 0.575 in. (*a*) What is the effective a-c resistance $r$ of each conductor of this circuit? (*b*) What is the average a-c resistance in a three-phase circuit, flat spacing, of the same conductors with 0.575 inch between axes of the center conductor and each of the outside conductors?

*Solution* (*a*). From Table II, Appendix A: $d = 0.460$ inch, $R_{dc} = 0.0500$ ohm per 1000 feet; $d/s = 0.460/0.575 = 0.80$; $\mathbf{x} = 0.0277\sqrt{400/0.05} = 2.48$; $t/d = 0.5$. Also, from Part I of Fig. 4(*b*), $\mathbf{x} = 2.48$.

From Fig. 4(*a*) or from Part II of Fig. 4(*b*) with $\mathbf{x} = 2.48$ and $t/d = 0.5$, $A = 0.496$. From [33], or from Part III of Fig. 4(*b*), with $A = 0.496$ and $s/d = 0.8$, $R''/R'$(approx.) $= 1.21$. The correcting factor $K_c$ from Table IV, corresponding to $\mathbf{x} = 2.48$, $t/d = 0.5$, and $d/s = 0.8$, is $K_c = 0.982$. From [39], $R''/R' = 0.982 \times 1.21 = 1.19$. From Example 2, $R'/R_{dc} = 1.17$. Substitution in [19] gives $r = 0.050 \times 1.17 \times 1.19 = 0.0696$ ohm per 1000 feet.

*Solution* (*b*). From solution (*a*), $A = 0.496$ and $d/s = 0.80$. $A(d/s)^2 = 0.318$. From [35] or from Part IV of Fig. 4(*b*), $F_{\text{ave}} = 1.067$. From [38], $R''/R'$(three-phase) $= 1.067 \times 1.19 = 1.27$. The average a-c resistance of the three conductors of the three-phase circuit from [19] is

$$r = 0.0500 \times 1.17 \times 1.27 = 0.0743 \text{ ohm per 1000 feet}$$

**Example 7.** A 60-cycle single-phase circuit operated at 70°C with conductors described in Example 5 has a distance of 4 inches between axes of conductors. What is the effective a-c resistance of each conductor?

*Solution.* From Example 5, $R_{dc} = 0.0125$ ohm per 1000 feet, $d = 1.346$ inches, $t/d = 0.29$, and $R' = 0.0129$ ohm per 1000 feet.

From [31], $\mathbf{x} = 0.0277\sqrt{60/0.0125} = 1.92$; $d/s = 1.346/4 = 0.337$. With $d/s = 0.337$, $K_c = 1.000$. From Fig. 4(*a*), with $\mathbf{x} = 1.92$ and $t/d = 0.29$, $A = 0.32$. From [33] or Part III of Fig. 4(*b*), with $A = 0.32$ and $d/s = 0.337$, $R''/R'$(approx.) $=$ $R''/R' = 1.019$. The effective a-c resistance $r$ of each conductor is

$$r = R'' = R' \times \frac{R''}{R'} = 0.0129 \times 1.019 = 0.0131 \text{ ohm per 1000 feet}$$

**Proximity-Effect Inductance Ratios for Single-Phase Circuits of Identical Non-Magnetic Conductors.** Figure 2 gives the skin-effect internal-inductance ratio $L_i'/L_i$ in solid non-magnetic conductors versus $\mathbf{x}$ when $\mu = 1$, or $\mathbf{x}$ is defined in [28]. The inductance to neutral in a single-phase circuit with skin effect included, defined in [21], is applicable when the conductors are far enough apart for the non-uniform current distribution in the conductors to remain symmetrical about their axes. In this case the proximity-effect inductance ratio is unity. As the spacing between conductors is decreased, inductance to neutral $L''$ defined in [22] decreases.

H. L. Curtis[11] has given equations from which $L''$ may be calculated. These equations involve series which he states (when interpreted in terms of $\mathbf{x}$ defined in [28]) are not convergent for values of $\mathbf{x}$ higher than 4, approximately. As they converge slowly for lower values of $\mathbf{x}$, their application is laborious. Fortunately, a complete set of calculations is given for a single-phase circuit of No. 2 copper wires at six different distances between conductors and four frequencies which range from 500 to 3000 cps. Test results are also given. The maximum discrepancy between these observed and calculated values is 2% at $f = 3000$ cps and $d/s = 0.94$. At other and lower values of

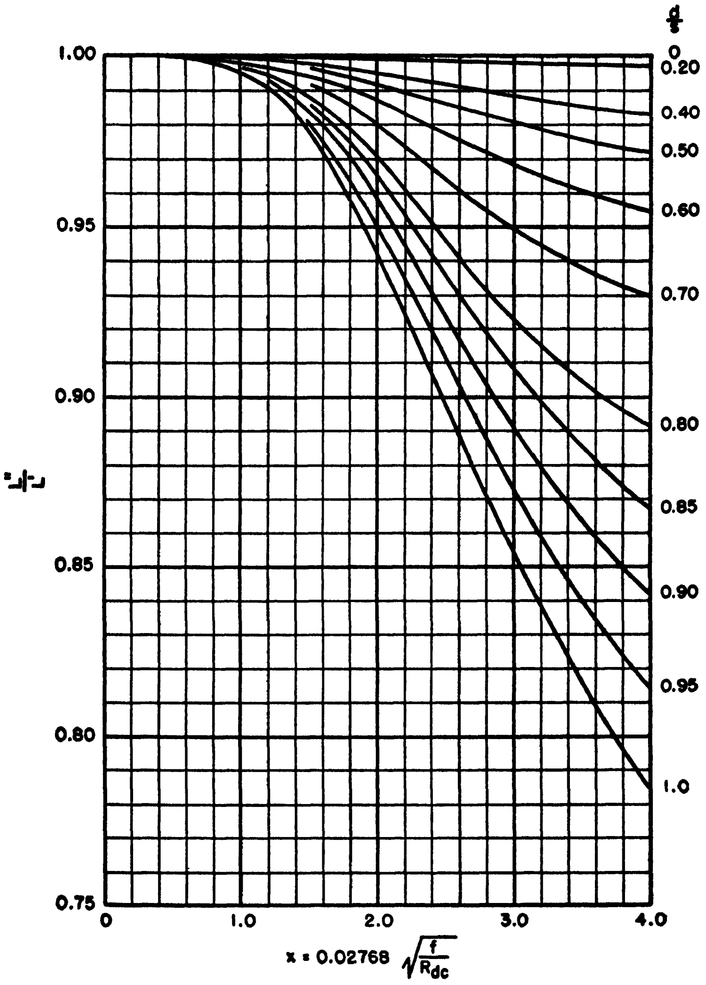

FIG. 5. Proximity-effect inductance ratios in single-phase circuits of solid non-magnetic round conductors. (H. L. Curtis, reference 11.)

$d/s$, the maximum discrepancy is 0.9%, and the average discrepancy of twenty-three observations is 0.4%.

The given equation shows that $L''/L'$ is a function of $x$ as defined in [28] and $d/s$ the ratio of diameter to spacing between axes of conductors. Figure 5 has been prepared by means of cross plots and interpolation from the calculations given by the author. In Fig. 5, $\mathbf{x}$ is abscissa, $L''/L'$ is ordinate, and $d/s$ is parameter, where $\mathbf{x}$ and $L''/L'$ are defined in [28] and [22], respectively.

**Example 8.** (*a*) What is the impedance to neutral in the single-phase circuit described in Examples 2 and 6 under normal balanced operation? (*b*) What would this impedance be, if uniform current distribution is assumed in the conductors?

*Solution* (*a*). From Example 6, $\mathbf{x} = 2.48$, $d/s = 0.80$, $R'' = 0.0696$ ohm per 1000 feet. From Fig. 5, $L''/L' = 0.945$. From Example 2, $x'_i = 0.0352$. From [18], $x' = 2\pi fL' = [0.0529\ (400/60) \log_{10} 2.50 + 0.0352] = 0.175$ ohm per 1000 feet. From [22], $x'' = 2\pi fL'' = 0.175 \times 0.945 = 0.165$. The impedance to neutral, with skin effect and proximity effect upon both resistance and reactance taken into account, is $Z = 0.0696 + j0.165$ ohm per 1000 feet.

*Solution* (*b*). If skin effect and proximity effect are neglected, current distribution in the conductors will be uniform. From [18] with $s/d = 1.25$, $Z = 0.050 + j(0.140 + 0.0383) = 0.050 + j0.178$ ohm per 1000 feet.

In the above example in which the frequency is 400 cps, the effective a-c resistance is increased 39% by skin effect and proximity effect, while the reactance to neutral is decreased only 8%. At 60 cps, both of these effects would be less. As the effect of non-uniform current distribution upon reactance is always less than upon resistance, and at 60 cps this effect is relatively small, the reactance corresponding to uniform current distribution is ordinarily used for 60-cycle circuits of non-magnetic conductors in non-magnetic media. In the case of effective a-c resistance, which affects power loss, a higher degree of precision is required.

## SELF- AND MUTUAL IMPEDANCES OF CIRCUITS IN MAGNETIC MEDIA

If a circuit consists of straight cylindrical parallel wires of non-magnetic material in a non-magnetic medium such as air, its positive-sequence self-impedance can be calculated as previously illustrated in this chapter without the necessity of considering the magnetic circuits involved. This simple procedure is not possible for electric equipment in which the flux paths are in iron, as when the conductors are coils with iron cores or are surrounded by iron. The relations between electric circuits and their associated magnetic circuits are summarized briefly below.

**Electric and Magnetic Circuits.** Current flowing in a closed circuit produces a *magnetomotive force*, determined by the ampere-turns of the circuit. Magnetomotive force produces magnetic flux. Magnetic flux or flux lines are closed loops which link the currents which produce them. Flux linkages with a circuit produced by current in the circuit are linkages of self-inductance. The number of linkages with the circuit depends upon the magnetomotive force and the *permeance* of the flux paths. Some of the flux lines which link the circuit may also link other circuits. The flux linkages mutual to two circuits are linkages of mutual inductance. The number of flux linkages with an electromagnetically coupled circuit depends upon the magnetomotive force producing the flux linkages and the permeance of the mutual flux paths. The *time rate of change* of the flux linkage with a circuit determines the voltage induced in the circuit. Inductances can be calculated or determined by test from the ratios of reactive induced voltages to the currents which produce them. The terms used here are defined below.

**Magnetomotive Force $F$.** The magnetomotive force (abbreviated mmf) around any path in gilberts is

$$F = 0.4\pi Ni \qquad [40]$$

where $N$ is the number of turns linked by the path and $i$ is the current in each turn in amperes. If the current is a constant d-c current, the mmf is constant. If the current is an alternating current, the mmf pulsates with the current and is in phase with it. If $i$ is instantaneous current, $F$ in [40] is instantaneous mmf.

**Flux Linkage $\psi$.** In a coil carrying current, there is always some flux which does not link all turns of the coil. The flux linkage $\psi$ in a coil of $N$ turns is the sum of the number of flux lines linking each turn of the coil. The average linkage per turn is $\psi/N$.

**Self- and Mutual Linkages.** When two coils are mutually coupled, part of the flux linking one coil will also link the other. Flux linkages are conveniently divided into self-linkages and mutual linkages. The self-linkage in a coil is the linkage due to current in that coil. Mutual linkage is the linkage in either coil due to current in the other coil. Leakage linkages in either coil do not link the other coil.

In addition to the flux linkages due to current, there may be linkages resulting from *residual magnetism*. When flux lines pass through iron, some of them remain after the magnetizing force is removed. The iron, in effect, becomes a permanent magnet of definite, but usually unknown, polarity. In dealing with problems where residual magnetism is important, the effects of maximum positive and maximum negative residual magnetism should both be considered.

The effect of *hysteresis* on flux linkages resulting from current depends upon whether the current producing the flux is increasing or decreasing. With alternating current, the effect of hysteresis is different in different parts of the cycle. These effects in many problems are small and can be neglected without serious error. Power losses because of hysteresis, however, may be more important. They are usually lumped with eddy-current losses in the iron and determined by test under specified conditions.

**Permeance $\mathcal{P}$.** Permeance depends upon the dimensions and permeability of the flux paths. For a path of length $\ell$, cross section $A$, and constant permeability $\mu$, the permeance $\mathcal{P}$ of the path is

$$\mathcal{P} = \frac{\mu A}{\ell} \tag{41}$$

The permeability $\mu$ of air is unity. The permeability of iron varies through a wide range. For any given sample, it varies with the degree of saturation in the iron. When flux paths are in iron and air in series, and dimensions and permeabilities are known, the permeance of each section of the path may be determined separately; total permeance is then the reciprocal of the sum of the reluctances (reluctance is the reciprocal of permeance) of the various portions of the total path.

Consider the two coils 1 and 2 of turns $N_1$ and $N_2$, carrying currents $i_1$ and $i_2$, respectively. The self-linkage $\psi_{11}$ in coil 1 at any instant due to the mmf $F_1$ of coil 1 is related to the permeance $\mathcal{P}_{11}$ of the coil 1 by the equation [12]

$$\frac{\psi_{11}}{N_1} = F_1\mathcal{P}_{11} \tag{42}$$

From [40] and [42],

$$\mathcal{P}_{11} = \frac{\psi_{11}}{0.4\pi N_1^2 i_1} \tag{43}$$

Likewise,

$$\mathcal{P}_{22} = \frac{\psi_{22}}{0.4\pi N_2^2 i_2} \tag{44}$$

The permeance of the mutual flux path $\mathcal{P}_{12} = \mathcal{P}_{21}$ may be expressed in terms of the flux linkage in one coil produced by the mmf of the other acting in the path of mutual flux:

$$\frac{\psi_{21}}{N_2} = F_1'\mathcal{P}_{21} \quad \text{or} \quad \frac{\psi_{12}}{N_1} = F_2'\mathcal{P}_{12} \tag{45}$$

where the first subscript of $\psi$ refers to the circuit in which linkages are produced by current in the circuit indicated by the second subscript, and $F_1'$ and $F_2'$ are used instead of $F_1$ and $F_2$ to indicate that the mmf's which produce mutual linkages may not be the same as those producing self-linkages. In closely coupled coils on the same iron core, $F_1'$ and $F_2'$ are substantially equal to $F_1$ and $F_2$, respectively. If $F'$ in

[45] is replaced by $F$ from [40], for *closely coupled circuits,*

$$\mathcal{P}_{21} = \mathcal{P}_{12} = \frac{\psi_{21}}{0.4\pi N_1 N_2 i_1} = \frac{\psi_{12}}{0.4\pi N_1 N_2 i_2} \qquad [46]$$

**Self- and Mutual Inductances of Static Circuits.** If $L$ is the self-inductance of the circuit in henries, $i$ the instantaneous current in the circuit in amperes, and $\psi$ the self-linkage due to the current $i$, then at any instant

$$\psi = Li \times 10^8 \qquad [47]$$

where $L$ may be a function of $i$.

If saturation and hysteresis are negligible, $L$ is independent of $i$. Then, expressed in henries, the self-inductance $L_{11}$ of any circuit 1 is

$$L_{11} = \frac{\psi_{11}}{i_1} \times 10^{-8} = 0.4\pi N_1^2 \mathcal{P}_{11} \times 10^{-8} \qquad [48]$$

The mutual inductance $L_{12}$ between two circuits may be defined as the ratio of the flux linkage in one circuit produced by current in the other to that current. Expressed in henries, with negligible saturation and hysteresis,

$$L_{12} = L_{21} = \frac{\psi_{21}}{i_1} 10^{-8} = 0.4\pi N_1 N_2 \mathcal{P}_{12} \times 10^{-8} \qquad [49]$$

From [48] and [49], the self-inductance of a static circuit is directly proportional to the square of the number of turns, the mutual inductance between two circuits to the product of the turns in the two circuits. Inductance is directly proportional to permeance and, like permeance, is a circuit parameter which is constant in a static circuit when linear relations exist between current and flux linkages.

**Voltage of Self-Inductance.** By Faraday's law, when the linkages of a closed circuit change, a voltage is induced in the circuit equal to $10^{-8}$ times the rate of *decrease* of linkages with respect to time. In volts, the induced voltage $e_i$ is

$$e_i = -\frac{d\psi}{dt} 10^{-8} = -p\psi 10^{-8} = -p(Li) \qquad [50]$$

where $p = d/dt$, the derivative with respect to time $t$.

If linear relations exist between current $i$ in a circuit and linkage $\psi$ with the circuit resulting from the current $i$, $L$ is constant and [50] becomes

$$e_i = -L\frac{di}{dt} = -L(pi) \qquad [51]$$

where $i$ is in amperes, $L$ is in henries, $e_i$ is an induced voltage *rise* in the positive direction of current flow. (See Volume I, page 16.) The voltage *drop* in the positive direction will have an opposite sign.

In [51], let the current be sinusoidal; $i$ may then be expressed by the equation

$$i = I_m \cos \omega t = I_m \cos 2\pi ft \qquad [52]$$

where $I_m$ is the crest or maximum value of the instantaneous current $i$, $f$ is frequency in cycles per second, and $t$ is time in seconds. Substitution of [52] in [51] gives

$$e_i = -L\frac{d}{dt}(I_m \cos 2\pi ft) = (2\pi fL)I_m \sin 2\pi ft \qquad [53]$$

In [53], $2\pi fL$ is the reactance $x$ corresponding to frequency $f$. In vector notation,

$$E_i = -jxI \qquad [54]$$

where $I$ and $E_i$ are rms current and induced voltage, respectively, and the induced voltage rise $E_i$ in the direction of current flow is 90° in time phase behind the current ($\sin \omega t$ lags $\cos \omega t$ by 90°); but the voltage *drop* in the circuit caused by the current $I$ is 90° ahead of the current.

If [54] is compared with [51], it will be seen that [54] can be derived from [51], when the current is sinusoidal, without differentiation if $p = d/dt$ is replaced by $j2\pi f$, $e_i$ by $E_i$, and $i$ by $I$.

If a voltage $e_t$ is applied to the terminals of a circuit having resistance $r_1$ ohms and self-inductance $L_{11}$ henries in series, which are independent of $i$, and positive direction of current flow is *into* the circuit, the induced voltage opposes the applied voltage. The total voltage rise in the same direction in the circuit, closed by the applied voltage, is zero; or the applied voltage is just equal to the voltage drop in the circuit. Therefore,

$$e_t - r_1 i - L_{11} p i_1 = 0 \qquad [55]$$

or

$$e_t = r_1 i_1 + L_{11} p i \qquad [56]$$

When the applied voltage $e_t$ is sinusoidal and the self-inductance constant, the current will also be sinusoidal, and [56] may be expressed in terms of rms voltages and currents if $p = d/dt$ is replaced by $j2\pi f$. Then, in vector notation,

$$E_t = (r_1 + j2\pi fL_{11})I_1 = (r_1 + jx_{11})I_1 = Z_{11}I_1 \qquad [57]$$

where $E_t$ and $I_1$ denote rms terminal voltage and current in circuit 1, respectively, and $Z_{11}$ is the self-impedance of circuit 1,

**Voltage of Mutual Inductance.** The voltage $e_{21}$ in volts induced in circuit 2 by current $i_1$ in circuit 1, when linear relations exist between current and flux linkages, is

$$e_{21} = \pm \frac{d}{dt}\psi_{21} \times 10^{-8} = \pm p\psi_{21}10^{-8} = \pm L_{21}pi_1 \qquad [58]$$

With sinusoidal current flowing in circuit 1, the voltage induced in circuit 2 is sinusoidal. When $p = d/dt$ is replaced by $j2\pi f$, and $e_2$ and $i_1$ by $E_2$ and $I_1$, respectively,

$$E_2 = \pm j2\pi fL_{21}I_1 = \pm jx_{21}I_1 \qquad [59]$$

where $I_1$ and $E_2$ are rms current in circuit 1 and induced voltage in circuit 2, respectively, $x_{21}$ is the mutual reactance between the circuits, and the sign ± indicates that the voltage $E_2$ may be positive or negative. When the circuits are parallel, $E_2$ is a positive voltage drop in the positive direction of $I_1$.

**Self-Impedance Determined by Test.** With current as reference vector, the voltage $E_t$ in [57] consists of a component $I_1r_1$ in phase with the current and a reactive component $I_1x_{11}$ which is 90° out of phase with the current. When $L_{11}$ is constant, and a sinusoidal voltage of fundamental frequency is applied between circuit terminals, the fundamental-frequency voltage drop in the circuit in the direction of current flow is equal to the applied voltage. The fundamental-frequency reactance $x$ is the ratio of the reactive component of this voltage drop to the current; similarly, the resistance of the circuit is the ratio of the in-phase component of this voltage drop to the current.

In circuits where saturation is appreciable, harmonic currents will be present when a sinusoidal voltage is applied across circuit terminals. These currents result from induced harmonic voltages which are short-circuited by the source of the applied voltage. In such circuits, it is convenient to define fundamental-frequency reactance in terms of applied fundamental-frequency current and the reactive voltage of fundamental frequency due to this current.

The fundamental-frequency reactance $x$ of a static circuit is the ratio of the fundamental-frequency component of reactive voltage drop in the circuit due to the fundamental-frequency component of current to this component of current. By this definition, fundamental-frequency reactance $x$ will vary with the degree of saturation; and it is constant only at no saturation.

The equations and definitions given here are further discussed and applied in the chapters which follow.

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# CHAPTER III

## ELECTRICAL CHARACTERISTICS OF INSULATED CABLES

The methods used in Volume I, Chapters XI and XII, to determine the positive-, negative-, and zero-sequence inductive impedances and capacitive admittances of overhead transmission lines, and in Chapter VI to derive the equivalent Π or *T* of a line with distributed constants, can also be applied to cables. As with overhead transmission lines, cables are completely defined when circuit dimensions and materials are known. Because of the many types of cables, relatively long descriptions may be required before circuit dimensions, if not explicitly given, can be determined. The term cable is used in this chapter in its inclusive sense, signifying the manufactured product. A brief discussion of the types of cables used in power systems will be given here; for more detailed information, see Bibliography at the end of this chapter.

**Types of Cables.** Cables are classified as underground, submarine, aerial, depending upon location. They are classified by their protective finish. The finish may be metallic: for example, a lead sheath; it may be non-metallic: for example, weatherproof braid. Cables may be equipped with armor, of which there are several types, depending upon location and use. They are classified according to the type of insulation. The insulation materials most widely used are compounds of rubber, varnished cambric, and impregnated paper. There are various grades of rubber compounds, and three types of impregnated paper insulation: solid, oil-filled, and gas-filled. Cables are classified as single-conductor, two-conductor, three-conductor, and so on, depending upon the number of conductors per cable; as shielded (type H) or non-shielded (belted), depending upon the presence or absence of metallic shields over the insulation. In multiconductor belted cable, each conductor is separately insulated; these insulated conductors are then cabled (twisted about each other), the interstices rounded out with fillers, and the whole wrapped with a belt of additional insulation, over which is placed the protective finish. See Fig. 1. The belted cable with non-metallic finish may be shielded at the higher voltages by applying a shielding tape over the belt and under the finish. In multiconductor shielded cable, each insulated conductor is wrapped with a thin metallic shielding tape (usually copper, 3 to 5 mils thick). They are then cabled together with fillers, bound together by a binder tape, and provided with a protective finish. See Fig. 2. The binder tape may be magnetic (steel) but is usually non-magnetic (copper or bronze) or non-metallic (rubberized tape, fibrous tape). In the type H cable, the shielding tape gives a radial configuration to the electrostatic field and also aids in conducting heat from the body of the cable to the sheath or other finish.[1,2]

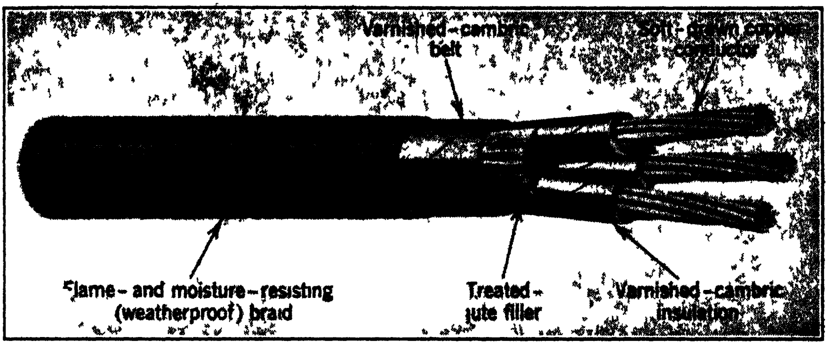
Varnished-cambric belt
Soft-drawn copper conductor
Flame- and moisture-resisting (weatherproof) braid
Treated jute filler
Varnished-cambric insulation

Fig. 1. Belted cable.

**Standards and Specifications.** The Insulated Power Cable Engineers Association (IPCEA) has prepared " General Specifications for Wire and Cable with Rubber and Rubber-Like Insulation,"[3] and " Specifications for Varnished Cambric Insulated Cables,"[4] which have been approved by the National Electrical Manufacturers Association (NEMA) as adopted standard or recommended practice. The Association of Edison Illuminating Companies (AEIC) has prepared " Specifications for Impregnated Paper Insulated, Lead Covered Cable, 'Solid' Type,"[5] and "Specifications for Impregnated Paper Insulated, Lead Covered Cable, 'Oil Filled' Type."[6] They have also prepared " Specifications for Impregnated-Paper-Insulated, Lead Covered Cable, ' Low-Pressure Gas-Filled ' Type."[7] These standards and specifications give approved conductors, maximum allowable copper temperatures, and minimum thicknesses of insulation of a given type for various rated cable voltages in grounded and ungrounded systems. They also give maximum allowable power factor of a given type of insulation, approved thicknesses of lead sheath and of shielding tape and binder tape, and specify tests to be met on dielectric strength and bending. Tables and other information from these standards and specifications are included in this chapter and in Appendix A.

The purpose of these specifications is to insure the furnishing of cable of high quality. As specifications are frequently revised, it is reasonable to assume that all *installed* cables do not necessarily conform to specifications written after their installation.

The *rated voltage of a cable* is its rated phase-to-phase voltage. " Oil-filled "[6] and " solid-type "[5] paper insulated cables have a normal operating tolerance of 5% above rated voltage for continuous operation and 10% above rated voltage during emergencies of not more than 15 minutes' duration. Varnished cambric insulated cables[4] have an emergency operating tolerance of 5% above rated voltage. Emergency overvoltage should not exceed a total time of more than 300 hours in any one year. In addition a short-time overvoltage tolerance of 10% above the rated voltage for not more than 15 minutes is allowed. For cables with rubber and rubber-like insulation,[3] the permissible maximum voltage is the same as the rated voltage.

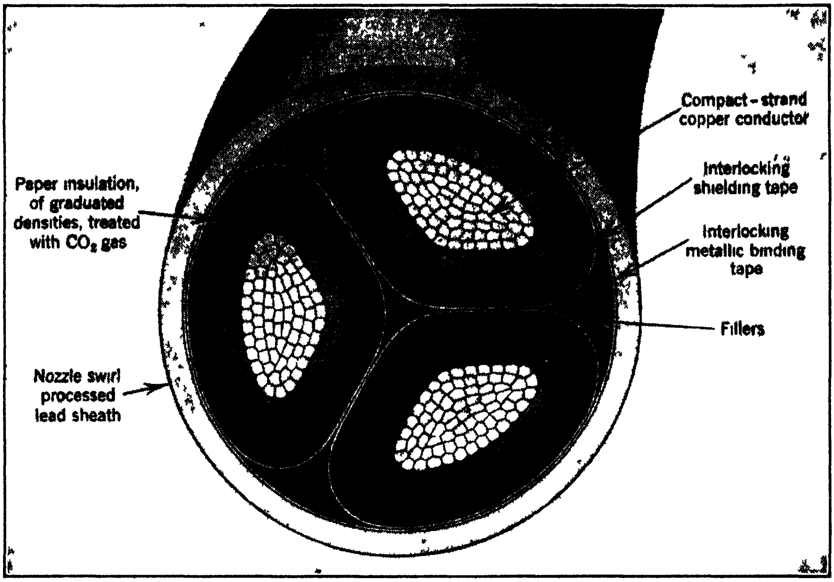
Compact-strand copper conductor
Interlocking shielding tape
Interlocking metallic binding tape
Fillers
Paper insulation, of graduated densities, treated with $CO_2$ gas
Nozzle swirl processed lead sheath

FIG. 2. Type H cable.

**Grounded System.** Quoting from AEIC specifications:[5,6] "A system is considered to have a grounded neutral if the neutral is permanently grounded to earth and if facilities are provided to insure prompt isolation of a faulty element of the system." The tables of Appendix A at the higher voltages show less insulation for grounded than for ungrounded systems.

**Conductors.** Practically all cable conductors are of annealed copper —not hard-drawn as in overhead transmission circuits. Solid conductors are used in cables of small sizes, but comparative stiffness makes them undesirable for large cables. Their advantages over stranded conductors of the same current-carrying capacity are lower cost, smaller overall cable diameter, and less weight. Conductors smaller than No. 6 A.W.G. are usually solid; larger sizes are usually stranded. However, for special applications, solid conductors as large as 4/0 have been used.

The stranded conductors used in *fixed cable installations* are of two types, standard strand and compact strand. (For portable use, conductors of greater flexibility are usually required.) Compact strand is made by drawing a stranded conductor through rolls so that the separate strands are pressed closely together, thereby making a conductor of smaller cross section which requires less insulation of a given thickness and has a smaller overall cable diameter. Compact strand conductors are round or sector.

In three-conductor cable, the conductors are round (Fig. 1) or sector (Fig. 2). In single-conductor cables, they are round (including annular) or segmental. A segmental conductor consists of three or four separately insulated conductors operated in parallel. For the same voltage and load current, the segmental type of single-conductor cable requires less insulation volume and has a smaller overall cable diameter than the annular conductor. The conductors of two-conductor cables are round or D-shaped.

**Diameters and Resistances of Conductors.** Table I, Appendix A, is a comparative wire gauge table. Tables II and III, Appendix A, give outside diameters in inches and d-c resistances in ohms per 1000 feet at 25°C of solid and standard strand annealed copper conductors of 100% conductivity; Table IV gives inside and outside diameters and d-c resistances of annular conductors with rope centers. To find the d-c resistance $R_t$ at any temperature $t$, in centigrade degrees,

$$R_t = R_{25}[1 + 0.00385(t - 25)] \qquad [1]$$

where $R_{25}$ is the d-c resistance at 25°C read from Table II, III, or IV, Appendix A.

*Skin-effect resistance ratios* for frequencies of 25 and 60 cps at 65°C are given in Tables III and IV, Appendix A. These ratios vary with temperature; they can be determined at other temperatures and frequencies, if required, from Fig. 2 or 3 of Chapter II.

*Proximity-effect resistance ratios* can be obtained from Fig. 4 of Chapter II.

**Thickness of Insulation.** Tables V–XI, Appendix A, give approved thicknesses of insulation at various rated voltages (at date given) for the types of cable and cable insulation used in power systems.

**Characteristics of Insulation.** The types of insulation used in power cable will be discussed with reference to the following characteristics: dielectric strength, resistance to ionization, resistance to heat, specific inductive capacity, leakance and dielectric loss, mechanical flexibility, and resistance to moisture. The effects of these characteristics on the maximum voltage rating of insulation will then be discussed. For any given cable installation, as in the selection of other equipment, where there is a choice between two or more types of insulation, either one of which will give satisfactory service, the most economical one is chosen, economy being based on yearly operating costs and length of life as well as on initial cost.

**Dielectric Strength.** The dielectric strength of a material is defined as the critical field intensity or voltage gradient at which breakdown of the material occurs; it is customarily expressed in terms of volts per mil or kilovolts per millimeter. Dielectric strengths of samples of the same insulation vary with temperature, thickness of sample, duration of voltage application, and to a lesser extent with wave shape of applied voltage and the configuration of the electrostatic field. Comparison of the dielectric strength of different types of insulation should therefore be made under identical test conditions. Cable-insulating materials, in decreasing order of dielectric strengths, are paper, varnished cambric, and rubber.

Consider a section of single-conductor cable consisting of a round conductor, insulation, and sheath. With sinusoidal voltage applied across the insulation between the round conductor and the sheath, the flow of current through the insulation is radial, and cross sections of the equipotential surfaces between conductor and sheath are circles. The variation of the voltage gradient along radial lines is non-uniform, being greatest at the conductor and least at the sheath. An increase in insulation thickness will increase the disruptive critical voltage of the insulation, but will not give a *proportional* decrease in voltage gradient at the conductor surface; it will not therefore allow a proportional increase in voltage rating. Determination of the minimum allowable conductor diameter for a given rated voltage is influenced by the potential gradient at the conductor surface. It will be noted in the tables of Appendix A that the minimum allowable diameter increases with an increase in rated voltage.

On the basis of dielectric strength alone, with a conductor of suitable diameter, the maximum voltage rating of a given type of insulation can be increased (but not in proportion) by increasing the thickness of the insulation until it becomes too bulky to warrant its use. Dielectric strength, however, is not the only factor that limits voltage rating.

**Resistance to Ionization.** A distinction is made between the voltage which causes a cable failure by disrupting the insulation and a voltage which produces ionization with possible deterioration of insulation. Ionization in cable insulation is local. It corresponds to corona at the surface of a conductor in an overhead transmission circuit; breakdown of cable insulation corresponds to flashover in air between conductors or between conductor and tower. Ionization in cable insulation is not ionization of the particles of the insulation itself, but of voids, gas, air, within the insulation, or of air at the insulation surface. Ionization of a particle occurs when the voltage stress across it is sufficient to disrupt it. Ionization is influenced by both heat and pressure, increasing with temperature and decreasing with increasing pressure. Ionization produces heat and, where air is ionized, ozone as well as heat. Ozone is very destructive to ordinary rubber compounds, but does not injure paper or varnished cambric; local high temperatures are also destructive to ordinary rubber compounds. Local heat in voids will damage impregnated paper or varnished cambric if it is sufficiently intense actually to burn the insulation in contact with the void or to raise the temperature of the surrounding insulation above its safe operating value. Deterioration, once started, may spread until breakdown occurs. It is important that the voltage rating and thickness of insulation be such as to reduce to a minimum the possibility of deterioration because of ionization.

The contractions and expansions of a cable during temperature changes produce voids in paper insulation of the solid type, or in varnished cambric. By limiting the maximum operating temperature and thus the temperature variation, the formation of voids is limited. Impregnated paper, oil-filled type, has the highest resistance to void formation of any type of insulation, because the oil under pressures higher than atmospheric prevents the formation of voids during temperature changes.

**Resistance to Heat.** The maximum approved copper temperatures (at date given) for the types of insulations used in power cables at various rated voltages are given in Table XII, Appendix A. These approved temperatures are based on the heating of the insulation due to load current in the conductors and on the effects of dielectric losses upon insulation temperatures. Although each type of insulation has a maximum approved copper temperature at rated cable voltage, some types of insulation deteriorate more rapidly than others if subjected to temperatures above these rated temperatures. Ordinary rubber compounds belong in this class.

**Specific Inductive Capacity.** The specific inductive capacity or the *dielectric constant* of any material (abbreviated S.I.C. and represented by $K$) is defined as the ratio of the capacitance of a condenser with the material as a dielectric to the capacitance of a similar condenser with air as the dielectric. For air, the S.I.C. is unity: $K = 1$. Table I gives the approximate S.I.C. of the types of insulation used in power cables with their power factors, which vary with temperature. Because of dielectric constants greater than unity and the smaller spacings used in cable circuits, the capacitances and charging currents of cables are very much higher than those of overhead transmission circuits of the same length at the same voltage.

TABLE I

APPROXIMATE SPECIFIC INDUCTIVE CAPACITIES AND POWER FACTORS OF INSULATION AT 60 CYCLES PER SECOND

| | Rubber | Varnished Cambric | Solid Paper | Oil-Filled | Gas-Filled |
|---|---|---|---|---|---|
| Approx. SIC ($K$) | 5 | 5 | 4.0 | 3.7 | 4.0 |
| Power factor | Not over 5% | | | | 1.5% |

POWER FACTOR IN PER CENT

| Temperature, Degrees C* | Varnished Cambric[4] | | Solid Paper[5] | Oil-Filled[6] |
|---|---|---|---|---|
| | Single Conductor and Shielded Cable | Multiconductor Belted Cable | 8 Kv and Over, Rated Volts, Phase-to-Phase | |
| 50 | 6.0 | 8.0 | | |
| 60 | 7.0 | 10.0 | 0.90 | 0.60 |
| 70 | 8.0 | 12.5 | 1.5 | 0.75 |
| 75 | 8.7 | 14.0 | 1.8 | |
| 80 | 9.6 | 16.0 | 2.1 | 0.90 |
| 85 | 11.0 | 18.0 | 2.5 | 0.97 |
| 90 | | | 3.0 | 1.05 |

* Uniform temperature throughout cable.

[4-6] See Bibliography.

**Leakance and Dielectric Loss.** Charging currents flowing through cable insulation have power components. These power components cause a leakance or dielectric loss. Above very low voltages, the dielectric loss at a given temperature, and no ionization, varies as the product of the leakance and the square of the applied voltage. At rated voltage and temperature, the dielectric loss $W$ per unit length of a balanced three-phase circuit is

$$W = 3|V_{a1}|^2 g_1 \qquad [2]$$

where $V_{a1}$ is positive-sequence voltage to neutral and $g_1$ is positive-sequence leakance to neutral. Let

$\cos\theta$ = power factor of insulation in per unit at given temperature

$C_1$ = positive-sequence capacitance to neutral in farads

$b_1 = 2\pi f C_1$ = positive-sequence capacitive susceptance to neutral in mhos

$Y_1$ = positive-sequence admittance to neutral in mhos

$g_1$ = positive-sequence leakance to neutral in mhos

Then

$$Y_1 = g_1 + jb_1 = g_1 + j2\pi f C_1 = \sqrt{g_1^2 + b_1^2}\,(\cos\theta + j\sin\theta) \qquad [3]$$

From [3],

$$g_1 = \sqrt{g_1^2 + b_1^2}\cos\theta \qquad [4]$$

Solving [4] for $g_1$,

$$g_1 = b_1 \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} = b_1 \cot\theta = (2\pi f C_1)\cot\theta \qquad [5]$$

Replacing $g_1$ in [2] by its value from [5], at rated temperature and voltage, the dielectric loss $W_1$ in a symmetrical three-phase circuit under balanced operation is

$$W_1 = 3|V_{a1}|^2 (2\pi f C_1)\cot\theta \qquad [6]$$

For practical values of insulation power factor (see Table I), $\cot\theta = \cos\theta$, approximately, and [6] becomes

$$W_1 = 3|V_{a1}|^2 (2\pi f C_1)\cos\theta \qquad [6a]$$

In [6] and [6a], if $V_{a1}$ is in volts and $2\pi f C_1$, the positive-sequence capacitive susceptance to neutral, is in mhos per 1000 feet, $W_1$ will be in watts per 1000 feet of three-phase circuit.

Equations [6] and [6a] developed for a three-phase circuit can be used to determine the positive-sequence dielectric loss under balanced conditions in a symmetrical single-phase or $n$-phase circuit if 3 is replaced by 2 for a single-phase circuit and by $n$ for an $n$-phase circuit, $V_{a1}$ and $C_1$ being positive-sequence voltage and positive-sequence capacitance to neutral, respectively.

**Alternate Formula.** The formula for dielectric loss given in the *Underground Systems Reference Book*[8] is

$$W = \frac{0.000106e^2fn^2K \cos \theta}{G} \text{ watts per foot of cable} \qquad [6b]$$

where $e$ = voltage to neutral in kilovolts
$f$ = frequency in cycles per second
$n$ = number of conductors ($n = 1$ for single conductor or multiconductor shielded cables)
$K$ = S.I.C. of insulation
$\cos \theta$ = power factor of insulation in per unit
$G$ = geometric factor, which is related to $C$ by the equation $G = 0.0169nK/C$
$C$ = capacitance in microfarads per 1000 feet of cable.

This formula, which checks [6*a*], also checks [6] for practical values of power factor. Values of positive-sequence capacitance to be used in [6], [6*a*], or [6*b*] may be obtained from Fig. 17 (explained later). See Example 7 for application of these equations. As the power factor of a given insulation increases with temperature (see Table I), dielectric losses also increase with temperature.

Low dielectric loss is desirable for the following reasons: (*a*) For a given type of insulation, quality varies inversely with dielectric loss. This criterion, however, does not hold when different types of insulation are compared; for example, when paper is compared with varnished cambric. (*b*) A low dielectric loss permits a cable to carry a higher load without overheating than it can carry if the loss is high. This is particularly important on high-voltage cable. (*c*) The dielectric loss, being in evidence 24 hours per day, produces an appreciable increase in the annual operating cost of the cable, especially at the higher voltages.

**Flexibility and Resistance to Moisture.** Rubber compounds have the highest mechanical flexibility and the greatest resistance to moisture; varnished cambric is second in both of these qualifications; paper is third. Paper insulation always requires a moisture-proof finish; varnished-cambric insulation may or may not, depending upon where it is used. Low-voltage rubber-insulated cables with jackets of rubber compound having a high abrasive resistance are frequently buried directly in the earth in locations where overhead wires are considered undesirable.

All cables must be flexible to a certain degree, as they are subjected to twisting and bending during installation, and also during service because of temperature changes. Installation of insulated cable requires care and skill, as the cable must not be bent more sharply than the minimum permissible bending radius either during or after installation.

## Maximum Voltage Rating of Insulation

**Paper-insulated cable** has the highest dielectric strength, the highest allowable temperature, and the lowest dielectric loss of any of the insulations used in power cables; it also has the highest voltage rating.

*Solid paper-insulated cable* is impregnated with a heavy viscous filling compound to limit migration but which may not always fill up all the voids which may be formed during temperature changes in service. These voids are small, but their dielectric strength is low since they are usually at low pressure. The thicknesses of insulation used with paper of the solid type are those required to reduce the possibility of failure due to the ionization of voids within the insulation to a minimum rather than those dictated by the dielectric strength of the material. As an increase in insulation thickness at high voltages does not warrant a proportional increase in voltage rating, physical dimensions become the limitation to the maximum voltage rating of solid paper above 69,000 volts for single-conductor cables or 46,000 volts for three-conductor cables.

*Oil-filled paper-insulated cable* is impregnated with a relatively thin liquid oil, maintained at a positive pressure at all times by means of reservoirs, which prevents the formation of voids. Without voids in insulation of a given thickness, a cable can be operated at a higher voltage and also at a higher temperature. Oil-filled cable has been used for voltages up to 138 kv for single-conductor cables, and 69 kv for three-conductor cables, and can be made available for voltages up to 230 kv for single-conductor cables and 115 kv for three-conductor cables — values which will probably be increased in the future. The minimum economic voltage for three-conductor oil-filled cable at present, under ordinary conditions, is from 33 to 40 kv; but such cable has been used for voltages as low as 15 kv, where space is important or where there are very wide temperature changes or very high temperatures, and as low as 27 kv, where reliability is essential.

In *gas-filled paper-insulated cable*,[9] the sheathed cable is first vacuum-dried and impregnated with a heavy oil. The surplus oil is then drained out and the drained space filled with dry nitrogen, maintained under pressure. As with oil, the gas keeps voids filled with fluid under pressure. There are three distinct types of gas-filled cable systems: low-pressure, medium-pressure, and high-pressure.

*Low-pressure gas-filled cable* employs an ordinary single lead sheath, and working gas pressure is thereby limited to an average of 12 lb per sq in. to prevent sheath creepage in service. The use of low-pressure gas-filled cable is limited to 40 kv and below. However, this voltage range covers the majority of cable installations in this country. Insulation thickness is less than for solid paper but greater than for oil-filled paper of the same rated voltage. The thickness of the lead sheath is comparable with that for oil-filled cable.

In *medium-pressure systems*, increased working gas pressure allows material reduction in insulation thickness. To obtain the necessary mechanical properties, a sheath of unusual strength is needed. At the present time, such a sheath is usually of reinforced construction, commonly used where excessive pressures are encountered in oil-filled systems. This means that it is possible safely to increase working gas pressure to 40 lb per sq in. or higher with a corresponding increase in voltage rating as compared with low-pressure systems operating at 12 lb pressure. For the same voltage rating the corresponding reduction in insulation thickness is about 30%. This particular type of medium-pressure system has an economic voltage range from 40 to 69 kv.

In *high-pressure systems*, at a gas pressure of 200 lb per sq in., the reduction in insulation thickness and increase in working voltage stress are on the same basis as those for oil-filled cable systems. Full advantage of high gas pressure operation can be safely realized by drawing the cable into a welded steel pipe line, this pipe line being buried directly in the earth. There is also available the self-contained high-pressure gas-filled cable with specially reinforced sheath suitable for 200 lb pressure, and designed for installation in ducts. In certain situations this self-contained type can be used more effectively than the pipe type.

**Varnished Cambric.** The dielectric loss in varnished cambric limits its maximum voltage rating, not only because of losses during normal temperature conditions, but also because of the possible effects of ionization of voids in the insulation. The increase in losses caused by ionization produces a rapid increase in power factor which results in increased losses and further increases in temperature. By limiting the voltage rating and the maximum permissible operating temperature, dielectric losses are limited to safe values.

**Rubber Compounds.** Rubber compounds are listed[3] as Code Grade, Intermediate Grade, 30% Grade, and higher grade special-performance compounds. The maximum voltage rating of Code Grade is limited by its dielectric strength. For the better grades, one limitation is ionization of air in voids which may exist at the surface of the conductor with the consequent destruction of the rubber. Another is the difficulty of applying the heavy walls of insulation required at the higher voltages. A third limitation is the difficulty of justifying economically the use of rubber insulation at the higher voltages.

Code, Intermediate, or other compounds of grades lower than 30% are not acceptable[3] for circuit voltages of 5001 volts and over. Special-performance rubber compounds which are heat-resistant, moisture-resistant, and ozone-resistant are listed in reference 3 (see Table V, Appendix A) for voltages up to 27,000 volts, but are rarely used above 15 kv.

## CABLE DIMENSIONS

The principal dimensions to be calculated in the determination of the electrical characteristics of cable circuits are distances between phase conductors and diameters of sheaths.

**Round Conductors.** The outside diameters of solid, standard strand, and annular round conductors are given in Tables II–IV, Appendix A. *Round conductors of compact strands* have diameters approximately 10% less than those of conductors of standard strands of the same circular mils of copper.

**Spacing between Phase Conductors.** For a *three-phase circuit of single-conductor cables* $a$, $b$, and $c$, the geometric mean distance $s$ between axes of conductors is

$$s = \sqrt[3]{s_{ab}s_{ac}s_{bc}}$$

where $s_{ab}$, $s_{ac}$, and $s_{bc}$ are the distances between the axes of the conductors indicated by the subscripts.

In a **three-conductor cable,** the spacings between conductors can be determined from the type of cable and the insulation thicknesses given in the tables of Appendix A. For **belted-type cable** with round conductors, $s$ in inches is

$$s = d + 2T \qquad [7]$$

For type H (shielded) cables with round conductors,

$$s = d + 2T + 2t' \qquad [8]$$

where $d$ = outside diameter of round conductor in inches (For conductors of compact strands, multiply $d$ of non-compact equivalent conductor by 0.9.)
$T$ = thickness of conductor insulation in inches
$t'$ = allowance for shielding tape in inches.

The equivalent spacing $s$ between sector conductors for use in calculating zero-sequence impedance may be evaluated from [9] after $D_i$ has been calculated, or it may be calculated from [7] or [8], if $d$ is multiplied by a factor[2] between 0.82 and 0.86. (See Example 4.)

Equations [7] and [8] apply to cables with paper or varnished-cambric insulation. The shielding tape used with paper insulation is always intercalated with paper, which gives an equivalent thickness of 8 mils, or 0.016 inch for $2t'$ in [8]. The shielding tape used with varnished cambric is 3 to 5 mils thick and may have a fibrous tape in addition for mechanical reasons, which gives approximately 8 mils, or 0.016 inch for the term $2t'$ in [8].

In rubber-insulated multiple-conductor cables, the belted construction is not used. In both shielded and non-shielded three-conductor rubber-insulated cables, except at very low voltages, rubberized tape of 10 mils minimum thickness is used to protect the insulation. To include the effect of overlap, add $2 \times 15$ mils $= 0.030$ inch to $s$ in [7], and use $2(15 + 10)$ mils $= 0.050$ inch in [8] for $2t'$.

**Core Diameter of Three-Conductor Cables.** The calculated diameter of a three-conductor cable under the sheath, called the core diameter, is also the calculated inside diameter $D_i$ of the sheath.

With *round conductors*, $D_i$ can be expressed in terms of $s$, the spacing between phase conductors, from the geometry of a three-conductor cable. Multiplying $s$ in [7] or [8] by $[1 + (2/\sqrt{3})]$ and adding twice the thickness of the belt for a belted cable and twice the thickness of the binding tape for a type H cable, $D_i$ in inches is then increased by 3% to compensate approximately for manufacturing tolerances:

$$D_i = 1.03(2.155s + 2t) \qquad [9]$$

where $t$ is thickness of belt or binding tape. The thickness of the belt is obtained from the tables of Appendix A. For paper or varnished cambric type H cable, $12\frac{1}{2}$ mils is allowed for binding tape, giving $2t = 0.025$ inch. With rubber-insulated cable, the binding tape is the same as that used over the conductor insulation; allowing for overlap, $2t$ in [9] $= 0.030$ inch.

For *sector conductors of standard strands* (non-compact), $D_i$ in inches is approximately*

$$D_i = 1.03[2\sqrt{A} + 4.5(T + t') + 2t] \qquad [10]$$

For *sector conductors of compact strands*, $D_i$ in inches is approximately*

$$D_i = 1.03[1.85\sqrt{A} + 4.44(T + t') + 2t] \qquad [10a]$$

where $A$ = cross section of sector conductor in circular mils × $10^{-6}$
$T$ = thickness of conductor insulation in inches
$t'$ = allowance for shielding tape in inches for type H (shielded) cable (For belted cables, $t' = 0$.)
$t$ = thickness of belt or allowance for binding tape in inches.

Equations [9]–[10*a*] give calculated values of $D_i$ increased by 3% to compensate approximately for manufacturing tolerances; they should not be regarded as exact. The recommended thicknesses of conductor insulation and belt, given in the tables of Appendix A, are minimum approved thicknesses. When actual thicknesses are known, these known values may be substituted in the above equations with the factor 1.03 omitted.

**Outside Sheath Diameter.** A pure lead sheath (99.85% lead[4,5]) is the standard finish for underground paper or varnished-cambric-insulated cables in ducts, and is sometimes used with rubber-insulated cables. Pure lead is susceptible to some forms of corrosion. It is liable to be damaged by electrolytic action, and it may fail because of crystallization under vibration. An alloy in the lead (usually 2% tin) makes the sheath more resistant to crystallization. Alloy sheaths are used in locations where resistance to crystallization is essential.

The minimum thickness of lead sheath for various core diameters $D_i$ is given in Table II for the general types of insulation.[3-6] The outside diameter $D_o$ of the sheath is

$$D_o = D_i + 2L \qquad [11]$$

or

$$D_o = D_i + 2L(1.03) \qquad [11a]$$

where $D_i$ is given by [9] for a three-conductor cable with round conductors and by [10] or [10*a*] for sector conductors. $L$ is the thickness of the lead sheath in inches. The factor 1.03 in [11*a*], as in [9]–[10*a*], may be applied to $2L$ to compensate approximately for manufacturing tolerance. When $D_i$ and $L$ are given, [11] should be used.

---

* Empirical formula; sector lay-off determines constants which differ somewhat in sector conductors made by different manufacturers.

TABLE II

Recommended Thickness of Lead Sheath

| Rubber[3] and Varnished Cambric[4] | | |
|---|---|---|
| Core Diameter | Thickness of Lead Sheath | |
| *Inches* | *64th Inch* | *Mils* |
| 0 –0.425 | 3 | 47 |
| 0.426–0.700 | 4 | 63 |
| 0.701–1.050 | 5 | 78 |
| 1.051–1.500 | 6 | 94 |
| 1.501–2.000 | 7 | 109 |
| 2.001–3.000 | 8 | 125 |
| 3.001 and larger | 9 | 141 |

| Solid Paper[5] | | Oil-Filled Paper[6] | |
|---|---|---|---|
| Core Diameter | Thickness of Lead Sheath | Core Diameter | Thickness of Lead Sheath |
| *Inches* | *Mils* | *Inches* | *Mils* |
| 0 –0.462 | 75 | 0 –1.459 | 110 |
| 0.463–0.648 | 80 | 1.460–1.644 | 115 |
| 0.649–0.833 | 85 | 1.645–1.830 | 120 |
| 0.834–1.018 | 90 | 1.831–2.015 | 125 |
| 1.019–1.203 | 95 | 2.016–2.200 | 130 |
| 1.204–1.388 | 100 | 2.201–2.385 | 135 |
| 1.389–1.574 | 105 | 2.386–2.570 | 140 |
| 1.575–1.759 | 110 | 2.571–2.756 | 145 |
| 1.760–1.944 | 115 | 2.757–2.940 | 150 |
| 1.945–2.129 | 120 | 2.941–3.126 | 155 |
| 2.130–2.314 | 125 | 3.127–3.311 | 160 |
| 2.315–2.499 | 130 | 3.312–3.496 | 165 |
| 2.500–2.685 | 135 | 3.497–3.681 | 170 |
| 2.686–2.870 | 140 | 3.682–3.867 | 175 |
| 2.871–3.055 | 145 | 3.868–4.052 | 180 |
| 3.056–3.240 | 150 | | |
| 3.241–3.425 | 155 | | |
| 3.426–3.611 | 160 | | |
| 3.612–3.796 | 165 | | |
| 3.797–3.981 | 170 | | |
| 3.982–4.166 | 175 | | |

[3-6] See Bibliography.

**D-C Sheath Resistance.** The d-c resistance $r_s$ of the lead sheath at any temperature $t$°C can be calculated from [2] and [7] of Chapter II when the resistivity $\rho$ and constant-mass temperature coefficient of resistance $\alpha$ are known. Assuming a resistivity $\rho_0$ at 0°C of $20.4 \times 10^{-6}$ ohm (cm cube) and a temperature coefficient $\alpha_0$ at 0°C of 0.00411, $r_s$ at 40°C is

$$r_s = \frac{\rho_0 \ell_0}{A_0}[1 + (40 - 0)\alpha_0] \qquad [12]$$

where $\ell_0$ is length and $A_0$ is area of cross section at 0°C. Neglecting change in dimensions with temperature, or assuming that the calculated values of $\ell$ and $A$ apply at 0°C, $r_s$ in ohms per 1000 feet at 40°C is

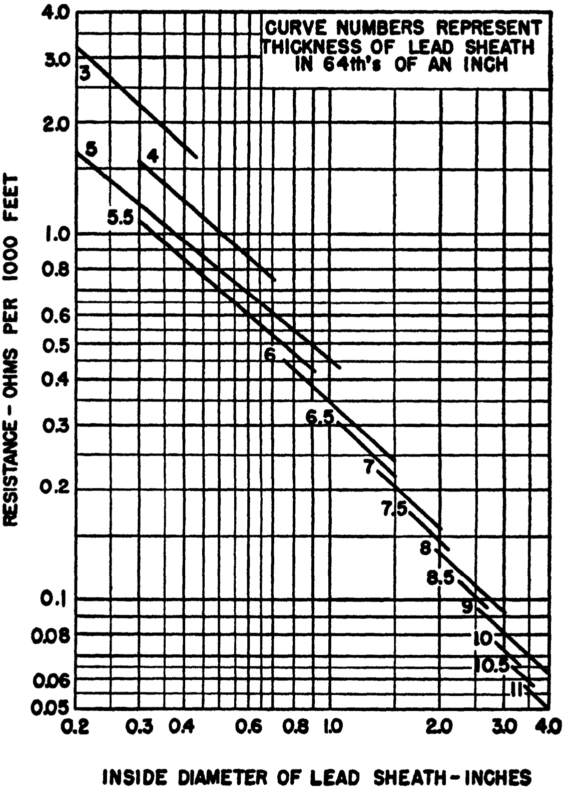

FIG. 3. Resistance of lead sheath at 40°C. (Resistivity of lead = 20.4 × 10$^{-6}$ ohm (cm cube), and constant-mass temperature coefficient = 0.00411 at 0°C.)

$$r_s = \frac{20.4 \times 10^{-6} \times 12{,}000}{2.54(D_o^2 - D_i^2)\frac{\pi}{4}}[1 + 40(0.00411)] = \frac{0.143}{D_o^2 - D_i^2} \qquad [12a]$$

where $D_o$ and $D_i$ are outside and inside diameters, respectively, of the sheath in inches.

For any given core diameter $D_i$ and sheath thickness in sixty-fourths of an inch, $r_s$ in ohms per 1000 ft at 40°C can be read directly from Fig. 3. Calculations were made from [12*a*], the thicknesses of lead given in Table II and $D_0$ given by [11] being used. If [11*a*] is used for $D_0$ instead of [11], $r_s$ will be approximately 97% of its value read from Fig. 3.

## POSITIVE- AND NEGATIVE-SEQUENCE IMPEDANCES

The equations developed in Volume I and summarized in Chapter I of this volume for the positive- and negative-sequence self-impedances ($Z_{11}$ and $Z_{22}$) and mutual impedances ($Z_{12}$, $Z_{21}$, etc.) between the sequence networks of unsymmetrical static circuits are applicable to cables. See [7]–[12] and [59]–[63] of Chapter I.

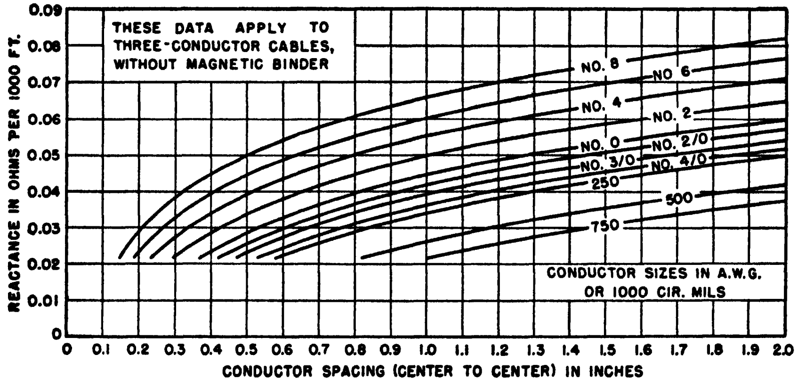

FIG. 4. Single-conductor, standard strand, and three-conductor cables without magnetic binder (for small spacings). Ohms per 1000 feet to neutral, at 60 cycles.

Equation [59] of Chapter I, which gives the positive- or negative-sequence self-impedance of a *three-phase circuit* of identical parallel cylindrical conductors of outside diameter $d$ and geometric mean spacing $s$ between conductors at any frequency $f$, is directly applicable to a three-phase circuit of single-conductor or three-conductor cables if uniform current distribution in the conductors is assumed. In ohms per 1000 feet,

$$Z_{11} = Z_{22} = r_a + jx = r_a + j\left(0.0529 \frac{f}{60} \log_{10} \frac{2s}{d} + x_i\right) \qquad [13]$$

where $r_a$ and $x_i$ are resistance and internal reactance, respectively, of one conductor $a$, and $x$ is total reactance to neutral. The internal reactance $x_i$ for tubular conductors is given in Fig. 1, Chapter II; for solid conductors and conductors of concentric stranding, $x_i$ is given in Table I, Chapter II. For the large conductors of many strands used in power cables, $x_i$ approaches that of the solid conductor; and [13] in ohms per 1000 feet becomes, approximately,

$$
\begin{aligned}
Z_{11} = Z_{22} &= r_a + jx \\
&= r_a + j\frac{f}{60}\left(0.0529 \log_{10} \frac{2s}{d} + 0.00575\right) \qquad [13a]
\end{aligned}
$$

As explained in Chapter II, skin effect may increase $r$ and decrease $x_i$; proximity effect may further increase $r$ and decrease $x$ the total reactance to neutral. Tables II–IV, Appendix A, give skin-effect resistance ratios at 65°C for frequencies of 25 cps and 60 cps; for other temperatures and frequencies, Figs. 2 and 3 of Chapter II may be applied.

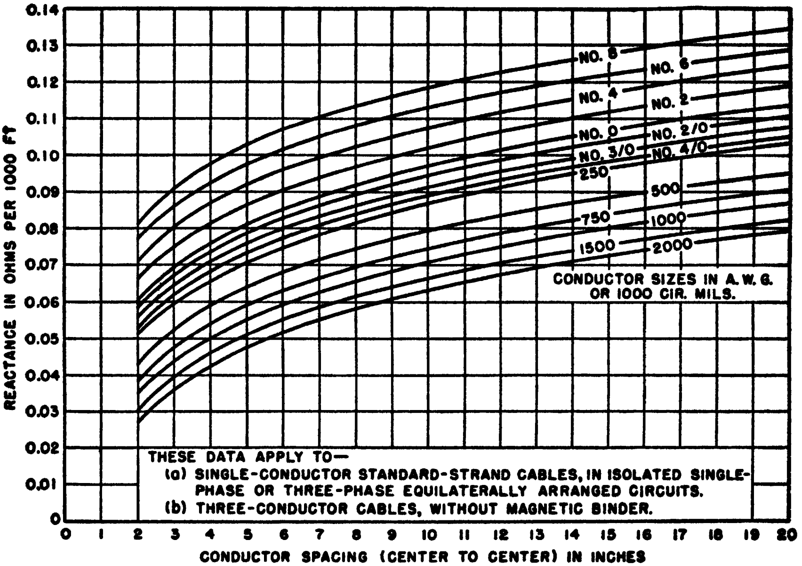

FIG. 5. Single-conductor, standard strand, and three-conductor cables without magnetic binder (for large spacings). Ohms per 1000 feet to neutral, at 60 cycles.

Equation [13], under the stated assumptions, also gives the positive-sequence impedance to neutral in a *single-phase* or a *two-phase* circuit, analyzed by positive- and zero-sequence symmetrical components (Volume I, page 297).

**Positive- or Negative-Sequence Self-Reactances from Curves.** Unless a high degree of precision is required, the influence of skin effect and proximity effect upon *reactance* may be neglected, and positive-sequence reactance to neutral (also negative-sequence, in a three-phase circuit) may be calculated from [13] or read from Figs. 4–6. Figures

4 and 5 apply to round conductors of standard concentric stranding in single-conductor or three-conductor cable. Figure 6 applies to single-conductor annular cables. The internal reactance $x_i$ is included in these curves.

For **sector conductors,** the reactance is approximately that determined from Fig. 4 or Fig. 5 for a round conductor of the same cross section and the same insulation thickness, multiplied by the corresponding correcting factor of Table III.

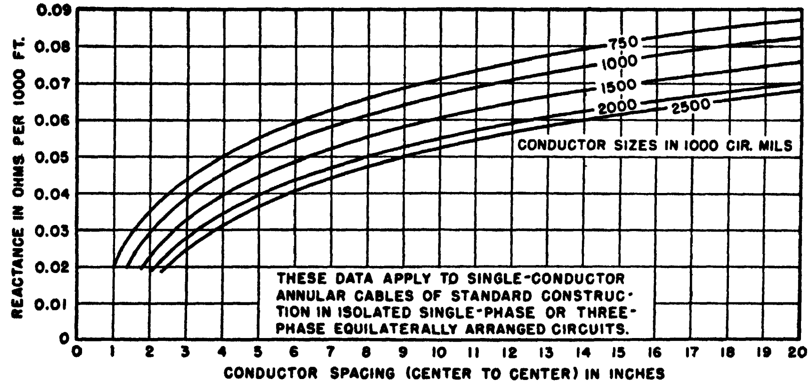

Fig. 6. Single-conductor, annular cables. Ohms per 1000 feet to neutral, at 60 cycles.

**Three-Conductor Cable with Lead Sheath.** In a three-conductor cable, the phases are completely transposed so that the circuit is essentially a symmetrical one. Positive-sequence currents flowing in the conductors induce no resultant voltages in the negative- or zero-sequence networks; nor do they induce voltages in parallel circuits. It is standard practice to ground the sheaths of three-conductor cables at each manhole and at their terminals. With positive-sequence currents only in the conductors, no resultant longitudinal voltage will be induced in the sheath, and no resultant longitudinal currents will flow. However, eddy currents will flow in the sheath (whether grounded or ungrounded) because the three conductors are not equidistant from individual points of the sheath. Eddy currents in the sheath increase the effective a-c resistance of the cable. Shielding tape or non-magnetic binding tape in contact with the sheath has slight influence on positive-sequence resistance, but magnetic binding tape increases the positive-sequence *reactance.* Table III gives correcting factors for three-conductor cables with lead sheaths to be applied to the d-c resistances of the conductors given in Appendix A.

## TABLE III

### 60-CYCLE CORRECTING FACTORS FOR THREE-CONDUCTOR CABLES*

| Conductor Size, A.W.G. or 1000 Cir Mils | D-C Resistance, Ohms per 1000 Feet at 70°C | Positive-Sequence A-C Resistance Factor | Positive-Sequence Reactance Factors | |
|---|---|---|---|---|
| | | | Magnetic Binder | Sector Conductors |
| 8 | 0.765 | 1.00 | 1.26 | |
| 6 | 0.482 | 1.00 | 1.26 | |
| 4 | 0.303 | 1.00 | 1.26 | |
| 2 | 0.191 | 1.01 | 1.26 | |
| 1 | 0.152 | 1.01 | 1.26 | |
| 0 | 0.120 | 1.02 | 1.26 | |
| 00 | 0.0952 | 1.03 | 1.26 | |
| 000 | 0.0755 | 1.04 | 1.26 | |
| 0000 | 0.0598 | 1.05 | 1.26 | |
| 250 | 0.0506 | 1.06 | 1.26 | .... |
| 300 | 0.0423 | 1.07 | 1.26 | 0.971 |
| 350 | 0.0362 | 1.08 | 1.26 | .... |
| 400 | 0.0317 | 1.10 | 1.25 | 0.962 |
| 500 | 0.0254 | 1.13 | 1.25 | 0.952 |
| 600 | 0.0211 | 1.16 | 1.24 | 0.935 |
| 700 | 0.0181 | 1.19 | 1.24 | 0.930 |
| 750 | 0.0169 | 1.21 | 1.23 | 0.926 |

* The correcting factors in this table were derived from reference 1.

These correcting factors include proximity effect, eddy-current losses in the sheath, and stray a-c losses in addition to skin effect; dielectric losses are not included. Correcting factors to be applied to the reactances read from Fig. 4 or Fig. 5 to include the effect of a magnetic binder in three-conductor type H cables are also given in Table III. The correcting factors in Table III are empirical, evaluated from test data;[1] they should therefore be regarded as approximate.

**Example 1.** Find the positive-sequence self-impedance of a 38.3-kv, three-conductor shielded cable, with 500,000-cm sector conductors and solid paper insulation, at an assumed copper temperature of 70°C, with (*a*) non-magnetic binder, (*b*) steel binder.

*Solution.* From Table III, $r = 0.0254 \times 1.13 = 0.0287$ ohm per 1000 feet. From Table IX, Appendix A, insulation thickness is 375 mils. From [8], for round conductors,

$$s = d + 2T + 2t' = 0.815 + 2(0.375) + 0.016 = 1.581 \text{ inches}$$

From Fig. 4, with $s = 1.581$,

$$x_{11} = 0.0370 \text{ ohm per 1000 feet}$$

(*a*) Applying the reactance correcting factor 0.952 for sector conductors from Table III, the positive-sequence self-impedance, with non-magnetic binder, is

$$Z_{11} = 0.0287 + j0.0352 \text{ ohm per 1000 feet}$$

(*b*) With magnetic binder, $x_{11}$ with non-magnetic binder is increased by the factor 1.25 from Table III:

$$Z_{11} = 0.0287 + j0.440 \text{ ohm per 1000 feet}$$

**Single-Conductor Cables with Open-Circuited Lead Sheaths.** Cable circuits are usually of such low impedance compared to the impedance of connected apparatus that any physical dissymmetry which may exist in the cable circuit is very unlikely to produce an electrical unbalance of any but the smallest magnitude. For those few cases in which the impedance of the cable circuit is a governing factor, or where an extremely high degree of precision is required, the method of symmetrical components may be applied to advantage. It should be remembered, however, that in cases where impedances met by phase currents are independent of their sequence, or where zero-sequence currents are absent, the use of the self- and mutual impedances of the conductors (Chapter I, [52]–[56]) may give simpler solutions than can be obtained by the method of symmetrical components. This point is illustrated in the development of [18].

Equations for unsymmetrical three-phase circuits in terms of *symmetrical components*, developed in Volume I, Chapter VIII, are summarized in Chapter I of this volume. Equations [7]–[9] express the positive-, negative-, and zero-sequence voltage drops in a three-phase circuit in the direction of current flow in terms of $I_{a1}$, $I_{a2}$, $I_{a0}$ and the sequence self- and mutual impedances of the circuit. $Z$ in these equations represents impedance, the first numerical subscript referring to the sequence of the voltage and the second to the sequence of the current producing the voltage.

In three-phase circuits of identical single-conductor cables with *sheaths open-circuited* and no current-carrying elements except the phase conductors, the positive- and negative-sequence self-impedances are given by [13] of this chapter. The mutual impedances between the sequence networks are given by [60]–[63] of Chapter I. When the cables are located at the vertices of an equilateral triangle, $s_{ab} = s_{ac} = s_{bc}$, and there are no mutual impedances between the sequence networks.

When the cables are *unsymmetrically spaced*, the dissymmetry produces mutual coupling between the sequence networks. Positive-sequence currents in the circuit therefore induce negative- and zero-sequence series voltages in the circuit between its terminals. These voltages, when impedances are finite, cause negative- and zero-sequence currents to flow in the unsymmetrical circuit and the connected system. In a system with negligible capacitance, ungrounded or grounded at one point only, there will be no zero-sequence currents unless the unsymmetrical circuit is part of a closed loop in the zero-sequence network. In a cable circuit during normal operation, the negative- and zero-sequence phase currents will be small. Neglecting their effects upon induced voltages, the *series voltages* $v_{a2}$ and $v_{a0}$ induced in the negative- and zero-sequence networks between the terminals of the unsymmetrical circuit are

$$\begin{aligned} v_{a2} &= I_{a1}Z_{21} \\ v_{a0} &= I_{a1}Z_{01} \end{aligned} \qquad [14]$$

where $I_{a1}$ is the positive-sequence current in the circuit, and $Z_{21}$ and $Z_{01}$ are defined in [61] and [63] of Chapter I in terms of circuit dimensions. The resulting negative- and zero-sequence currents in the circuit are, approximately,

$$\begin{aligned} I_{a2} &= \frac{v_{a2}}{z_{22}} = I_{a1}\frac{Z_{21}}{z_{22}} \\ I_{a0} &= \frac{v_{a0}}{z_{00}} = I_{a1}\frac{Z_{01}}{z_{00}} \end{aligned} \qquad [15]$$

where $z_{22}$ and $z_{00}$ are the series impedances in the negative- and zero-sequence networks, respectively, of the *system* to currents flowing in the unsymmetrical cable circuit. When the negative- and zero-sequence diagrams of the system are given, $z_{22}$ and $z_{00}$ can be calculated.

**Example 2.** In a three-phase 60-cycle circuit of single-conductor cables with sheaths open-circuited, the positive-sequence current is 100 amp; determine the negative- and zero-sequence voltages induced in the circuit between its terminals when one cable is equidistant from the other two in (*a*) a right-triangular arrangement, (*b*) a flat horizontal or vertical arrangement.

*Solution.* (*a*) Let conductor *a* be equidistant from conductors *b* and *c*. Then $s_{ab} = s_{ac}$, and $s_{bc} = \sqrt{2}s_{ab}$. The choice of the reference phase *a* is arbitrary, affecting the phase of the induced voltage with respect to the reference vector but not its magnitude.

From [61] and [63] of Chapter I, in ohms per 1000 feet of three-phase circuit,

$$Z_{21} = j0.0353\left(\frac{f}{60}\right)\log_{10}\frac{1}{\sqrt{2}} = -j\frac{f}{60}\,(0.00531)$$

$$Z_{01} = j\frac{f}{60}\,(0.00266)$$

From [14], with $I_{a1} = 100$ amp per phase,

$$|v_{a2}| = |I_{a1}Z_{21}| = 0.53 \text{ volt per 1000 feet}$$

$$|v_{a0}| = |I_{a1}Z_{01}| = 0.27 \text{ volt per 1000 feet}$$

(*b*) In a flat horizontal or vertical arrangement of cables and $s_{ac} = s_{ab}$, $s_{bc} = 2s_{ab}$; the mutual impedances and induced voltages will be twice those given in (*a*).

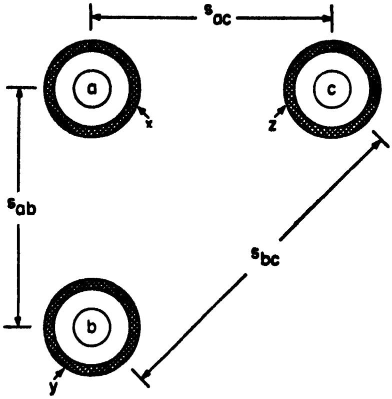

FIG. 7. Three-phase circuit of single-conductor cables. *a*, *b*, and *c* are the conductors; *x*, *y*, and *z* are their respective sheaths.

The negative- and zero-sequence currents resulting from these induced voltages can be determined when the negative- and zero-sequence diagrams of the system are known. The induced voltages are series voltages, not voltage to ground, and the resultant currents depend upon series impedances (Volume I, page 244).

**Voltages Induced in Sheaths of Single-Conductor Cables in Three-Phase Circuits during Normal Operation.** In Fig. 7, let *a*, *b*, *c* represent the phase conductors and *x*, *y*, *z* the sheaths enclosing conductors *a*, *b*, *c*, respectively. Let $Z$ with two subscripts, referring to conductor and sheath, indicate mutual impedance. With the sheaths on open circuit, the voltage drops $V_x$, $V_y$, and $V_z$ induced in sheaths *x*, *y*, and *z* in the direction of current flow by current $I_a$, $I_b$, and $I_c$ in the conductors, in terms of mutual impedances between conductors and sheaths, are

$$\begin{aligned} V_x &= I_aZ_{ax} + I_bZ_{bx} + I_cZ_{cx} \\ V_y &= I_aZ_{ay} + I_bZ_{by} + I_cZ_{cy} \\ V_z &= I_aZ_{az} + I_bZ_{bz} + I_cZ_{cz} \end{aligned} \qquad [16]$$

With *positive-sequence currents only* in the phase conductors, $I_a$, $I_b$, and $I_c$ in [16] are replaced by $I_{a1}$, $a^2I_{a1}$, and $aI_{a1}$, respectively, giving

$$\begin{aligned} V_x &= I_{a1}(Z_{ax} + a^2Z_{bx} + aZ_{cx}) \\ V_y &= I_{a1}(Z_{ay} + a^2Z_{by} + aZ_{cy}) \\ V_z &= I_{a1}(Z_{az} + a^2Z_{bz} + aZ_{cz}) \end{aligned} \qquad [17]$$

The mutual impedances in [17] have no resistance components; their reactance components are $2\pi f$ times their inductances. In Chapter I, the mutual inductance between two parallel non-concentric conductors is given by [56]; the mutual inductance between two concentric conductors is given by [57], and approximately by [58]. If [56] and [58] of Chapter I, where $D_o$ and $D_i$ are outside and inside sheath diameters, respectively, are substituted in [17], the induced sheath voltages in volts per 1000 feet of three-phase circuit are, approximately,

$$\begin{aligned} V_x &= jI_{a1}\,0.0529\,\frac{f}{60}\left(\log_{10}\frac{4\sqrt{s_{ab}s_{ac}}}{D_o + D_i} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right) \\ V_y &= ja^2I_{a1}\,0.0529\,\frac{f}{60}\left(\log_{10}\frac{4\sqrt{s_{ab}s_{bc}}}{D_o + D_i} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ab}}{s_{bc}}\right) \\ V_z &= jaI_{a1}\,0.0529\,\frac{f}{60}\left(\log_{10}\frac{4\sqrt{s_{ac}s_{bc}}}{D_o + D_i} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{bc}}{s_{ac}}\right) \end{aligned} \tag{18}$$

where $s$ is distance between axes of the conductors indicated by the subscripts (see Fig. 7).

The induced sheath voltages given by [18] apply to single-conductor cables in a three-phase circuit carrying positive-sequence currents when their sheaths are open-circuited and no currents can flow in them. In this case, it is unnecessary to determine the positive-, negative- and zero-sequence components of sheath voltages, the magnitude of the maximum voltage induced in any sheath being the important consideration.

**Symmetrical Components of Induced Sheath Voltages.** The three sheaths $x$, $y$, $z$ may be regarded as a three-phase circuit, parallel to the three-phase circuit consisting of conductors $a$, $b$, $c$. If the phase order in this circuit is $x$, $y$, $z$, and $x$, the sheath of conductor $a$, is selected as reference phase, the induced sheath voltages in [18] can be resolved into their symmetrical components $V_{x1}$, $V_{x2}$, and $V_{x0}$ by [4]–[6], Chapter I, $x$, $y$, $z$ replacing $a$, $b$, $c$, respectively, in the equations

$$\begin{aligned} V_{x1} &= jI_{a1}0.0529\,\frac{f}{60}\left(\log_{10}\frac{4s}{D_o + D_i}\right) \\ V_{x2} &= jI_{a1}0.0353\,\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right) \\ V_{x0} &= -jI_{a1}0.0176\,\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} + j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right) \end{aligned} \tag{19}$$

where $s = \sqrt[3]{s_{ab}s_{ac}s_{bc}}$, $I_{a1}$ is in amperes, and $V_{x1}$, $V_{x2}$, and $V_{x0}$ are the positive-, negative-, and zero-sequence induced sheath voltages, respectively, in volts per 1000 feet of three-phase circuit.

In a symmetrical circuit, $s_{ab} = s_{ac} = s_{bc}$. If this substitution is made in [19], $V_{x2} = V_{x0} = 0$. In a symmetrical circuit, the sheath voltages induced by positive-sequence currents are positive-sequence voltages. In an unsymmetrical circuit, there will be small components of negative- and zero-sequence induced voltages. For both symmetrical and unsymmetrical circuits, the positive-sequence induced voltage $V_{x1}$ is given approximately by the first equation in [19]. If [57] of Chapter I instead of [58] is used to determine [18] and [19], $V_{x1}$ is

$$V_{x1} = jI_{a1}\frac{f}{60}\left[0.0529\left(\log_{10}\frac{2s}{D_o} - \frac{D_i^2}{D_o^2 - D_i^2}\log_{10}\frac{D_o}{D_i}\right) + 0.0115\right] \quad [20]$$

where $I_{a1}$ is in amperes and $V_{x1}$ is in volts per 1000 feet of three-phase circuit. $V_{x2}$ and $V_{x0}$ are the same as given in [19]. The equations for $V_{x1}$ in [19] and [20] are given in slightly different form in references 2 and 10, respectively.

**Example 3.** In a three-phase 60-cycle circuit of single-conductor cable with dimensions in inches, $s_{ab} = s_{ac} = 6$; $s_{bc} = 6\sqrt{2}$; $D_i = 2$; $D_o = 2.25$. The sheaths are open-circuited. With positive-sequence currents only in the conductors, (*a*) what are the magnitudes of the voltages induced in the sheaths, and (*b*) what are the positive-, negative-, and zero-sequence induced sheath voltages in volts per ampere per 1000 feet of cable circuit? (*c*) With positive-sequence currents of 200 amp in the conductors of a circuit 5000 feet in length, determine the maximum sheath voltage and the positive-sequence sheath voltage.

*Solution.* (*a*) From [18], in volts per ampere per 1000 feet of three-phase circuit:

$$\left|\frac{V_x}{jI_{a1}}\right| = 0.0529\left(\log_{10}\frac{4 \times 6}{4.25}\right) = 0.0398$$

$$\left|\frac{V_y}{ja^2I_{a1}}\right| = 0.0529\left|\log_{10}\frac{24\sqrt[4]{2}}{4.25} - j\frac{\sqrt{3}}{2}\log_{10}\frac{1}{\sqrt{2}}\right| = |0.0438 + j0.0069| = 0.0444$$

$$\left|\frac{V_z}{jaI_{a1}}\right| = |0.0438 - j0.0069| = 0.0444$$

(*b*)

$$s = \sqrt[3]{6 \times 6 \times 6\sqrt{2}} = 6.74$$

From [19], the symmetrical components of sheath voltage in volts per ampere per 1000 feet of three-phase circuit are

$$\frac{V_{x1}}{I_{a1}} = j0.0529\left(\log_{10}\frac{4 \times 6.74}{4.25}\right) = j0.0424$$

$$\frac{V_{x2}}{I_{a1}} = j0.0353\log_{10}\frac{1}{\sqrt{2}} = -j0.0053$$

$$\frac{V_{x0}}{I_{a1}} = -j0.0176\log_{10}\frac{1}{\sqrt{2}} = j0.0027$$

if [20] is used instead of [19] to calculate $V_{x1}$,

$$V_{x1} = jI_{a1}\left[0.0529\left(\log_{10}\frac{2\times 6.74}{2.25} - \frac{4}{1.0625}\log_{10}\frac{2.25}{2}\right) + 0.0115\right]$$

$$\frac{V_{x1}}{I_{a1}} = j0.0424$$

For the degree of precision used here, there is no appreciable difference in this problem between $V_{x1}$ calculated from [19] and [20].

(*c*) In 5000 feet of circuit, if $I_{a1} = 200$ amp, the induced sheath voltages are

$$|V_y| = |V_z| = 200 \times 5 \times 0.0444 = 44.4 \text{ volts}$$

$$|V_{x1}| = 200 \times 5 \times 0.0424 = 42.4 \text{ volts}$$

**Positive-Sequence Mutual Impedance between Conductors and Sheaths.** In a three-phase circuit of single-conductor cables under normal operation, the phase currents are approximately positive-sequence currents. It has been shown that positive-sequence currents in the conductors induce voltages in the sheaths of single-conductor cables which are largely positive-sequence voltages in an unsymmetrical circuit, and entirely positive-sequence voltages in a symmetrical circuit.

Let $Z_{xa11}$ = positive-sequence mutual impedance between conductors $a$, $b$, $c$, and their sheaths $x$, $y$, $z$, treated as a three-phase circuit with phase order $x$, $y$, $z$. Then from [19], in ohms per 1000 feet, $Z_{ax11}$ is

$$Z_{ax11} = \frac{V_{x1}}{I_{a1}} = j0.0529\frac{f}{60}\log_{10}\frac{4s}{D_o + D_i} \tag{21}$$

If $Z_{ax21} = V_{x2}/I_{a1}$ and $Z_{ax01} = V_{x0}/I_{a1}$, from [19] in ohms per 1000 feet,

$$Z_{ax21} = \frac{V_{x2}}{I_{a1}} = j0.0352\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} - j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right)$$

[22]

$$Z_{ax01} = \frac{V_{x0}}{I_{a1}} = -j0.0176\frac{f}{60}\left(\log_{10}\frac{\sqrt{s_{ab}s_{ac}}}{s_{bc}} + j\frac{\sqrt{3}}{2}\log_{10}\frac{s_{ac}}{s_{ab}}\right)$$

Equations [21] and [22], when multiplied by $I_{a1}$, give the symmetrical components of voltage induced in the sheaths by positive-sequence currents in the conductors. As voltages are induced in the sheaths of single-conductor cables during normal operation whether the circuit is symmetrical or unsymmetrical, these voltages must be limited to safe values.

**Methods of Sheath Operation.** Sheaths of single-conductor cable may be operated (1) short-circuited or (2) open-circuited.

(1) If the sheaths are short-circuited, they are usually bonded and grounded at every manhole. This method reduces the sheath voltages to zero, but permits the flow of sheath currents.

(2) There are several methods of operating with the sheaths open-circuited. In each, sheath insulators which insulate the sheath sections from each other are introduced at every manhole or as often as necessary.

(*a*) When a ground wire is used, one terminal of each sheath section is bonded to the ground wire. The other terminal being open-circuited, no current can flow in the sheath. The voltage induced in each of the sheath sections during normal operation can be determined as described above.

(*b*) With cross bonding, connections are made across the sheath insulators between the sheaths of cables *a* and *b*, *b* and *c*, and *c* and *a*, so that the sheaths are transposed electrically, although the conductors are not. The sheaths may be bonded together and grounded at the end of each complete transposition. Voltages induced in the sheaths by positive-sequence currents add to zero in any complete transposition if the three lengths involved are equal; if the lengths are unequal, there will be small residual voltages which will produce sheath currents if solid bonding is used at the end of each complete transposition. Zero-sequence currents will flow in the sheaths exactly as they would in a system with short-circuited sheaths.

(*c*) The impedance bond consists of impedances introduced in each cable sheath across the sheath insulators or in the bond wires. This impedance tends to limit sheath currents to predetermined values without eliminating currents of any sequence.

(*d*) Bonding transformers[11] are small three-phase transformers connected Y-Δ; the Y terminals are connected to the insulated sections of cable sheaths at their ungrounded terminals; the neutral of the Y is grounded, and the Δ is operated on open circuit. Positive- and negative-sequence sheath currents meet the magnetizing impedance of the transformer and are thereby strictly limited in magnitude; zero-sequence currents meet the short-circuit impedance of the transformer since the Δ secondary acts as a short-circuited coil to currents of zero sequence.

**Skin Effect and Proximity Effect in Lead Sheaths.** With a given core diameter $D_1$, and thickness of lead $t$, $R_{dc}$, the d-c resistance of the sheath in ohms per 1000 feet, is given approximately by Fig. 3. The outside sheath diameter $D_o (= d)$, the ratio $t/d$, and $\sqrt{f/R_{dc}}$ can be calculated, and the skin-effect resistance ratio determined from Fig. 3

of Chapter II. At frequencies of 60 cps or less, it can readily be shown that the skin-effect resistance ratio is unity for the lead sheaths used in underground cable.

The positive-sequence proximity effect resistance ratio for a three-phase circuit consisting of three lead sheaths can be determined from Fig. 4(*b*) of Chapter II when $\mathbf{x}(=0.0277\sqrt{f/R_{dc}})$, $t/d$, and $d/s$ are known. At frequencies of 60 cps or less, the proximity-effect resistance ratio will not differ materially from unity for sheaths in separate ducts.

**Effective Positive-Sequence Resistance and Reactance of a Three-Phase Circuit of Single-Conductor Cables with Sheaths Short-Circuited.** The positive- or negative-sequence self-impedance of three sheaths in separate ducts, treated as a three-phase circuit, can be calculated from [13]. If $x_i$ in [13] is omitted and $d$ replaced by the average sheath diameter $(D_o + D_i)/2$, the impedance in ohms per 1000 feet is, approximately,

$$Z_{11(\text{sheath})} = r_s + j\left(0.0529\frac{f}{60}\log_{10}\frac{4s}{D_o + D_i}\right) = r_s + jx_s = Z_s \quad [23]$$

The d-c resistance $r_s$ of the sheath can be read directly from Fig. 3 in ohms per 1000 feet at 40°C and corrected to agree with the actual temperature and sheath resistivity, if known. As previously pointed out, skin effect is negligible in lead sheaths at frequencies of 60 cps or less; the effect of proximity upon resistance is also negligible when the sheaths are in separate ducts.

The positive-sequence self-reactance $x_s$ of the sheath in [23] and the positive-sequence mutual reactance $X_{ax11}$ between conductors $a$, $b$, $c$ and sheaths $x$, $y$, $z$ in [21] are the same. If $Z_{11(\text{sheath})} = Z_s = r_s + jx_s$, then $Z_{xa11} = jx_s$.

In a symmetrical circuit, let $I_{a1}$ and $I_{x1}$ be positive-sequence conductor and sheath currents, respectively. Let $V_{a1}$ be the positive-sequence voltage drop in the circuit *abc* in the direction of $I_{a1}$. If $Z_{11}$ is positive-sequence self-impedance of circuit *abc* with the sheaths open-circuited, then, with the sheaths short-circuited,

$$\begin{aligned} V_{a1} &= I_{a1}Z_{11} + I_{x1}Z_{xa11} = I_{a1}Z_{11} + I_{x1}(jx_s) \\ 0 &= I_{a1}Z_{xa11} + I_{x1}Z_s = I_{a1}(jx_s) + I_{x1}(r_s + jx_s) \end{aligned} \quad [24]$$

where $Z_{11}$ is given by [13] for the phase conductors, $Z_{xa11} = jx_s$ by [21], and $Z_s$ by [23]. Solving [24] for $I_{x1}$ and $V_{a1}$ in terms of $I_{a1}$,

$$\begin{aligned} I_{x1} &= -I_{a1}\frac{jx_s}{r_s + jx_s} \\ V_{a1} &= I_{a1}\left(Z_{11} + \frac{x_s^2}{r_s + jx_s}\right) = I_{a1}Z'_{11} \end{aligned} \quad [25]$$

The effective impedance $Z'_{11}$ of a symmetrical three-phase circuit of single-conductor cables with sheaths short-circuited and positive-sequence currents only in the phases is

$$\begin{aligned} Z'_{11} &= Z_{11} + \frac{x_s^2}{r_s + jx_s} = \left(r_a + \frac{r_s x_s^2}{r_s^2 + x_s^2}\right) \quad \left(x_{11} - \frac{x_s^3}{r_s^2 + x_s^2}\right) \\ &= r'_{11} + jx'_{11} \end{aligned} \qquad [26]$$

The *power loss P* in the symmetrical three-phase circuit under normal operation in watts per 1000 feet of three-phase circuit is

$$P = 3|I_{a1}|^2 r'_{11} = 3|I_{a1}|^2 \left(r_a + \frac{r_s x_s^2}{r_s^2 + x_s^2}\right) \qquad [27]$$

where $I_{a1}$ is in amperes and $r$ and $x$ are in ohms per 1000 feet. In an unsymmetrical circuit, the loss is somewhat greater because of negative- and zero-sequence sheath currents.

With the sheaths short-circuited, the effective a-c resistance of symmetrical cable circuits to positive-sequence currents during normal operation is increased from $r_a$ to $r'_{11}$, the value given in [26], and the effective reactance is decreased. The sheaths of single-conductor cable in separate ducts are short-circuited only when the load current is low; otherwise the losses may be excessive.

**Cables in the Same Duct.** When duct and cable dimensions permit, the single-conductor cables of a three-phase circuit may be drawn into the same duct. This is common practice in low-voltage circuits. The positive-sequence mutual impedance $Z_{xa11}$ between conductors and sheaths given by [21] varies with the log of the spacings between sheath centers. With the cables in the same duct, this mutual impedance, and therefore the induced positive-sequence sheath voltages, are lower than with the same cables in separate ducts. With smaller induced sheath voltages, and the partial neutralization of these voltages because of contact between sheaths, sheath currents and losses are very much less than when the same cables are placed in separate ducts and bonded.

## ZERO-SEQUENCE IMPEDANCES OF UNDERGROUND CABLES

Calculated zero-sequence impedances of overhead transmission lines show a satisfactory agreement with field tests when test values of earth resistivity are used in the zero-sequence impedance equations. Underground cables are influenced by the presence of miscellaneous underground or surface conducting materials, such as steel rails, water pipes, gas mains, subway structures. Theoretically, if the earth resistivity and the resistances, internal reactances, configuration, and dimensions of all cable elements, including miscellaneous conducting elements extraneous to the cable system, are known, calculation of the zero-sequence impedances of cables (although lengthy) could be accurately made. This information, however, is difficult and often impossible to secure. When the nature of the miscellaneous conducting material is unknown, accurate results can be obtained only by field tests. Even if a measurement is made today, it may be different a year from now, as subsurface construction of a conducting character is frequently being changed; for example, trolley tracks may be removed or water mains added or relocated. However, in cases when miscellaneous conducting material extraneous to the cables is unimportant in its effect, or its characteristics are known, the formulas given herein are applicable. Formulas will be given in a form convenient for evaluating the effects of miscellaneous conducting material. When extensive tests shall have been made on the installed cable systems throughout the country, empirical terms based on the location of the cables can be determined for use in these formulas. Such empirical terms have been determined and published for a large cable system.[12,13]

Equations are developed in Chapter XI, Volume I, for the zero-sequence self-impedances of overhead transmission circuits and the zero-sequence mutual impedance between parallel circuits, with and without ground wires, in terms of self- and mutual impedances of earth-return circuits. These equations are directly applicable to cable circuits with non-magnetic elements if grounded sheaths are regarded as ground wires. Sheaths not grounded, or grounded at one terminal only of insulated sheath lengths, may be disregarded, as no longitudinal currents can flow in them.

The equivalent circuits developed in Chapter VI, Volume I, to replace two or more parallel three-phase overhead circuits in the zero-sequence network in terms of the zero-sequence self-impedance of each circuit alone and the zero-sequence mutual impedance between the circuits, taken two at a time, can be applied to all types of three-phase cable circuits. With three-conductor cables, and symmetrical circuits of single-conductor cables, there are no mutual impedances between the sequence networks. In unsymmetrical circuits of single-conductor cables there are mutual impedances between the sequence networks which are usually unimportant, but can be evaluated from the equations of Chapter XI, Volume I, if required.

The equations developed for overhead transmission circuit will be rewritten in a form more convenient for cable calculations.

## Self- and Mutual Impedances of Earth-Return Circuits

In Volume I, Chapter XI, Carson's[14] and Rüdenberg's[15] equations for the self-impedance of a conductor with earth return and the mutual impedance between two conductors with common earth return are given and applied. Carson's equations, developed for overhead circuits, involve the height of the conductors above ground; in a later paper,[16] he shows that the variation of impedance with distance below the earth's surface is relatively unimportant. If it is assumed that the self- and mutual impedances with earth return of cable conductors and sheaths, at the usual distances below the earth's surface, are substantially the same as they would be at zero height above ground, Carson's equations with $h = 0$ are applicable to underground cables. Except for a small difference in the reactance components, equations for the self- and mutual impedances of earth-return circuit with $h = 0$ given by Carson are in agreement with those given by Rüdenberg which are independent of $h$, the height of the overhead conductor above ground or the distance of the insulated conductor below the earth's surface.

$Z_{aa-g}$, the self-impedance of any cylindrical conductor $a$ with earth return, and $Z_{ab-g}$, the mutual impedance between any two parallel non-concentric cylindrical conductors $a$ and $b$ with common earth return, are given by the equations

$$Z_{aa-g} = r_a + j(x_i)_a + R_g + jX_{aa-g} \qquad [28]$$

$$Z_{ab-g} = R_g + jX_{ab-g} \qquad [29]$$

where

$R_g$ = resistance of earth-return path

$= \pi^2 f$ abohms per cm = 0.095 ohm per mile

= 0.018 ohm per 1000 feet [30]

$X_{aa-g}$ = component of self-reactance with earth return external to conductor $a$

$$= \frac{f}{60}\left[0.0529 \log_{10} \frac{\sqrt{\rho}}{\left(\frac{d_a}{2}\right)\sqrt{f}} + 0.2335\dagger\right]$$

ohms per 1000 feet [31]

$X_{ab-g}$ = mutual reactance with common earth return between $a$ and $b$

$$X_{ab-g} = \frac{f}{60}\left(0.0529 \log_{10} \frac{\sqrt{\rho}}{s_{ab}\sqrt{f}} + 0.2335^*\right) \text{ ohms per 1000 feet} \qquad [32]$$

where $\rho$ = earth resistivity in ohms (meter cube)
$f$ = frequency in cycles per second
$d_a$ = diameter of conductor $a$ in inches
$s_{ab}$ = spacing between axes of conductors $a$ and $b$ in inches
$r_a$ and $(x_i)_a$ = effective a-c resistance and internal reactance, respectively, of conductor $a$.

The term conductor here includes sheaths, ground wires or any current-carrying circuit element which can be represented by an equivalent cylindrical conductor.

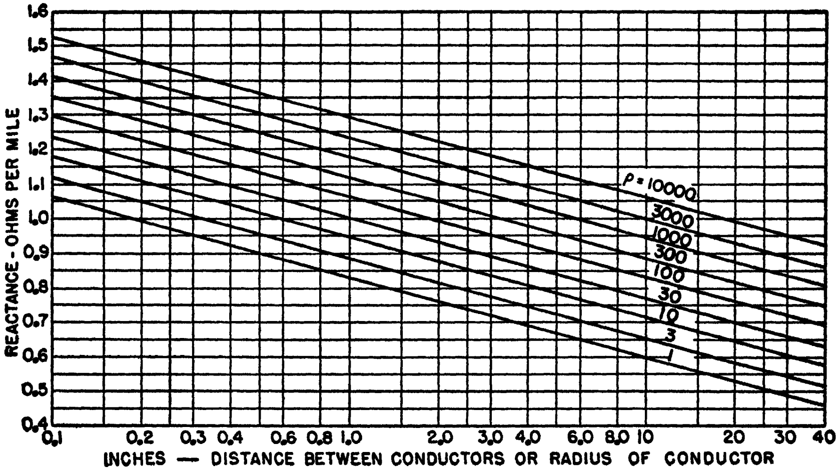

FIG. 8. Sixty-cycle mutual reactance between two conductors or sheaths with common earth return, or self-reactance of conductor or sheath with earth return if radius is used as abscissa. $\rho$ = earth resistivity in ohms per meter cube.

**Reactances of Earth-Return Circuits from Curves.** The self-reactance with earth return external to a conductor and the mutual reactance between two non-concentric conductors with common earth return at frequencies of 60, 50, and 25 cps are given in Figs. 8, 9, and

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[*] Carson's equations with $h = 0$ were used to derive $X_{aa-g}$ and $X_{ab-g}$ in [31] and [32]. If Rüdenberg's equations had been used, the constant term in [31] and [32] would be decreased by 0.0035 ($f$/60) ohms per 1000 feet.

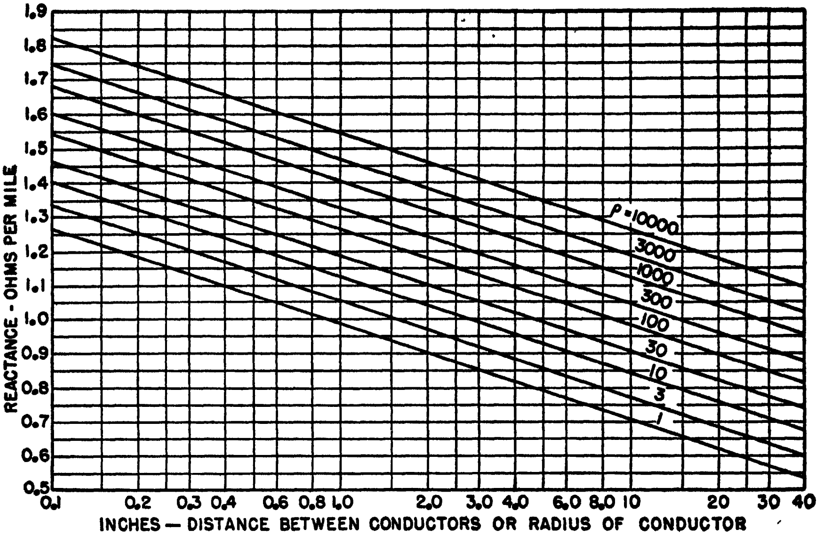

Fig. 9. Fifty-cycle mutual reactance between two conductors or sheaths with common earth return, or self-reactance of conductor or sheath with earth return if radius is used as abscissa. $\rho$ = earth resistivity in ohms per meter cube.

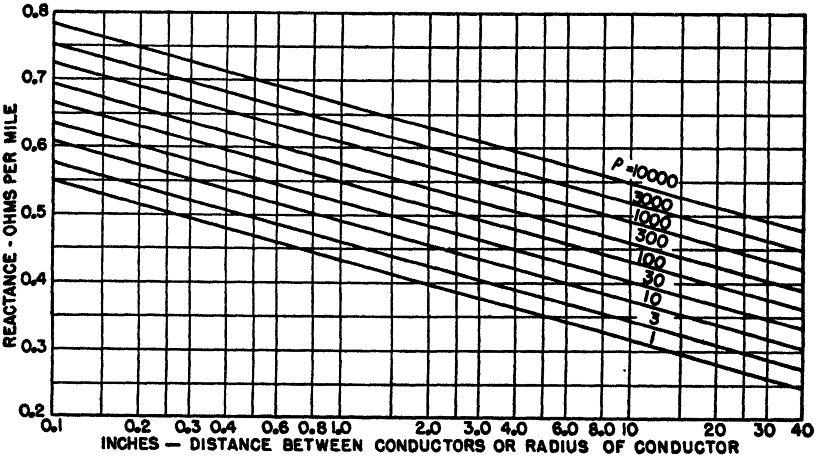

Fig. 10. Twenth-five-cycle mutual reactance between two conductors or sheaths with common earth return, or self-reactance of conductor or sheath with earth return if radius is used as abscissa. $\rho$ = earth resistivity in ohms per meter cube.

10, respectively. The reactances in these charts were calculated from Carson's equations[14] with $h = 0$. When determining the self-reactance with earth return external to the conductor, the abscissa is $d/2$, the radius of the conductor in inches. When determining the mutual reactance between two non-concentric conductors with common earth return, the abscissa is $s$, the spacing between the axes of the two conductors in inches. The ordinates are reactances in *ohms per mile.* These reactances must be divided by 5.28 to be expressed in *ohms per* 1000 *feet.*

**Lead Sheath with Concentric Inner Conductors.** The self-impedance with earth return of a lead sheath $w$ may be obtained from [28], [30], and [31] if $w$ replaces $a$ in [28] and [31]. The internal reactance $(x_i)_w$ corresponding to the flux in the lead is small (see Fig. 1, Chapter II) relative to the impedance of the earth-return path. Neglecting this internal reactance,

$$Z_{ww-g} = r_w + R_g + jX_{ww-g} \qquad [33]$$

The mutual impedance, with common earth return $Z_{aw-g}$ between sheath $w$ and a concentric inner conductor $a$, $b$, or $c$, can be evaluated by allowing $I_w$ to flow in sheath $w$ and return in the earth with no current in the conductors. In [33], $R_g + jX_{ww-g}$ is the component of self-impedance of $Z_{ww-g}$ external to the sheath. This component, which is the impedance of the earth-return path for current $I_w$, is also the component of mutual impedance with common earth return between $w$ and $a$, $b$, or $c$ *external to the sheath.* There is, in addition, a small component of mutual reactance $x_{aw}$ between $w$ and $a$, $b$, or $c$, produced by the flux in the lead with $I_w$ flowing in the sheath. With the flux in the lead neglected, or the sheath assumed to be infinitely thin (but retaining its resistance $r_w$), the following equation applies:

$$Z_{aw-g} = Z_{bw-g} = Z_{cw-g} = R_g + jX_{aw-g} = Z_{ww-g} - r_w = R_g + jX_{ww-g} \qquad [34]$$

where $R_g$ is given by [30] and

$$X_{aw-g} = X_{ww-g}$$

$$= \frac{f}{60}\left[0.0529 \log_{10} \frac{\sqrt{\rho}}{\left(\frac{d_w}{2}\right)\sqrt{f}} + 0.2335\right] \text{ohms per 1000 feet} \qquad [35]$$

$X_{aw-g} = X_{ww-g}$ can be calculated from [35] in ohms per 1000 feet or read from Figs. 8–10 in ohms per mile with $d_w/2$ the radius of sheath $w$ in inches as abscissa.

If it is required to include the internal self-reactance $(x_i)_w$ of sheath

$w$ and the mutual reactance $x_{aw}$ between $a$ and $w$ *internal* to the sheath, [33] and [34] become

$$Z_{ww-g} = r_w + R_g + jX_{ww-g} + (x_i)_w \qquad [33a]$$

$$Z_{aw-g} = R_g + j(X_{aw-g} + x_{aw}) = R_g + (X_{ww-g} + x_{aw}) \qquad [34a]$$

The internal self reactance $x_i$ is given by [55] of Chapter I, or Fig. 1 of Chapter II; the mutual reactance $x_{aw}$ internal to the sheath may be evaluated by subtracting [52] from [57] in Chapter I and multiplying by $2\pi f$. The internal mutual reactance $x_{aw}$ is slightly higher than the internal reactance $(x_i)_w$ because all the flux in the lead caused by current in the sheath links conductor $a$, but there are partial linkages with the sheath $w$. If $(x_i)_w$ and $x_{aw}$ are assumed to be equal, they can be evaluated by subtracting [52] from [58] in Chapter I and multiplying by $2\pi f$; then, approximately,

$$(x_i)_w = (x_{aw}) = 0.0529 \frac{f}{60} \log_{10} \frac{2D_o}{D_o + D_i} \text{ ohms per 1000 feet}$$

where $D_i$ and $D_o$ are internal and external sheath diameters, respectively, and $D_o$ is the same as $d_w$ used in [35]. If the value of $x_i = x_{aw}$, given above, is added to $X_{ww-g} = X_{aw-g}$ in [35] there results

$$(X_{aw-g} + x_{aw}) = [X_{ww-g} + (x_i)_w]$$

$$= \frac{f}{60}\left[0.0529 \log_{10} \frac{\sqrt{\rho}}{\frac{1}{4}(D_i + D_o)\sqrt{f}} + 0.2335\right] \text{ohms per 1000 feet} \qquad [35a]$$

Equation [35a] differs from [35] in that the average radius of the sheath is used instead of the outside radius. Equation [35] gives the component of self-reactance with earth return of sheath $w$ and the component of mutual reactance between $a$ and $w$ with common earth return, *external to the sheath.* With the flux in the lead caused by current $I_w$ in the sheath neglected, the reactance given by [35] is the total self-reactance of sheath $w$ and the total mutual reactance between $a$ and $w$ with earth return; [35a] gives these total reactances to a slightly closer approximation than [35].

In the work which follows, $Z_{ww-g}$ and $Z_{aw-g}$ as defined in [33] and [34] will be used. Except in the case of three-conductor cables with magnetic binders (where it is convenient to divide the zero-sequence self-impedance into components internal and external to the sheath) [33a]–[35a] will replace [33]–[35] if the average radius of the sheath is used instead of the outside radius in determining $Z_{ww-g}$ and $Z_{aw-g}$.

## Three-Conductor Lead-Covered Cables

In cables with *non-magnetic binders*, the copper shielding tape and non-magnetic binding tape in contact with the lead sheath carry so little current that their influence on zero-sequence impedances can be neglected. With a *magnetic binder*, zero-sequence currents flowing in the phase conductors induce voltages in the grounded sheath and the steel binder in contact with it which cause currents to flow in them. As the self-impedance of the lead sheath is very much lower than that of the steel binder, only a small portion of the total sheath and binder current will return in the binder. The binder as a path for zero-sequence currents can be disregarded, but as a path for flux it must be considered. Zero-sequence current returning in the sheath (assuming no current in the binder) will produce no flux in the binder because there is no flux within a hollow conductor carrying current. The flux external to the sheath produced by sheath current is not affected by the magnetic binder, but the flux within the sheath produced by zero-sequence currents in the conductors is affected.

R. L. Webb and O. W. Manz, Jr., in analyzing tests[12] made on the 27-kv, 60-cycle type H cables with steel binding tape of the Brooklyn Edison Company, separated zero-sequence self-impedances into their internal and external components, the periphery of the cable being the boundary between components. This method is presented in greater detail in a report[13] of the Edison Electric Institute and Bell Telephone System to which reference is made in their paper.

If zero-sequence impedance is separated into its internal and external components, the effect of the magnetic binding tape on internal impedance can be taken into account. Another advantage is that the effects of water pipes and gas pipes on the external impedance can be more easily evaluated. Also, where there are many sheaths in the same duct bank, calculations are simplified by this division.

**Notation.** In Fig. 11, let $a$, $b$, and $c$ indicate the phase conductors, $I_{a0}$ the zero-sequence current per phase, and $w$ the sheath of cable $abc$ whose zero-sequence self-impedance $Z_{00}$ is to be determined. Let $A$, $B$, and $C$ be the conductors and $v$ the sheath of a parallel cable; the zero-sequence mutual impedance $Z_{0m}$ between circuits $abc$ and $ABC$ is to be determined. Assume circuit $ABC$ to be grounded but open-circuited; then no current will flow in conductors $A$, $B$, and $C$, but voltages to ground will be induced in them by zero-sequence currents in circuit $abc$. Positive direction for *induced voltage drop* in circuit $ABC$ will be taken in the direction assumed as positive for $I_{a0}$. Let $u$ be the sheath of a third parallel cable. Other sheaths and miscellaneous conducting material not indicated in Fig. 11 may be present. Let $I_w$, $I_v$, and $I_u$ be the currents in sheaths $w$, $v$, and $u$, respectively, and $I_g$ the ground current. Let $p, \cdots n$ indicate additional sheaths, ground wires, or miscellaneous conducting elements, if present, and $I_p, \cdots I_n$ their respective currents. The zero-sequence self-impedance of the circuit *abc* and the zero-sequence mutual impedance between circuits *abc* and *ABC* will be represented by $Z_{00}$ and $Z_{0m}$, respectively, with additional subscripts to indicate the grounded sheaths. Thus, $Z_{00-wv}$ is the zero-sequence self-impedance with only grounded sheaths $w$ and $v$. In the three-conductor cables *abc* and *ABC* with sheath $w$ and $v$, respectively, the following relations exist between self- and mutual impedances (see [33] and [34]):

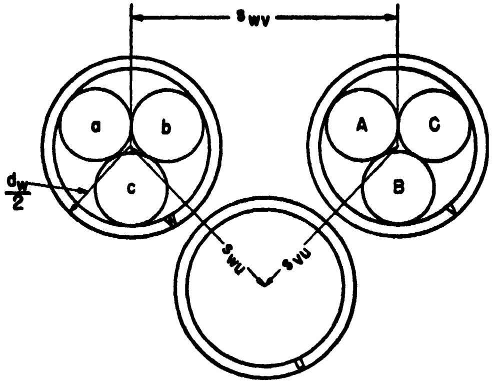

FIG. 11. Diagram indicating distances between sheath centers of cable *abc* with sheath *w*, cable *ABC* with sheath *v*, and an additional cable with sheath *u*.

$$
\begin{aligned}
Z_{ww-g} &= (r_w + R_g) + jX_{ww-g} = r_w + Z_{aw-g} \\
Z_{aw-g} &= Z_{bw-g} = Z_{cw-g} = R_g + jX_{aw-g} = R_g + jX_{ww-g} \\
&= Z_{ww-g} - r_w \\
Z_{vv-g} &= (r_v + R_g) + jX_{vv-g} = r_v + Z_{Av-g} \\
Z_{Av-g} &= Z_{Bv-g} = Z_{Cv-g} = R_g + jX_{Av-g} = R_g + jX_{vv-g} \\
&= Z_{vv-g} - r_v
\end{aligned}
\qquad [36]
$$

where $r_w$ and $r_v$ can be obtained from [12] or Fig. 3, $R_g$ from [30], and $X_{ww-g}$ $X_{aw-g}$ and $X_{vv-g}$ $X_{Av-g}$ from [35] or Figs. 8–10, corresponding to radius of the sheath $w$ or $v$ in inches.

The distance between axes of conductors $a$, $b$, $c$ and $A$, $B$, $C$ is also the distance between axes of sheaths $w$ and $v$; therefore, from [29],

$$Z_{aA-g} = R_g + jX_{aA-g} = Z_{wv-g} = Z_{av-g} = Z_{Aw-g} = R_g + jX_{wv-g} \qquad [37]$$

where the mutual reactances in [37] correspond to the distance $s_{wv}$ between the centers of sheaths $w$ and $v$ and can be read from Figs. 8 to 10. Equations corresponding to [37] can also be written for the mutual impedances between cables with sheaths $w$ and $u$, or $v$ and $u$, which correspond to distances $s_{wu}$ and $s_{vu}$, respectively. (See Fig. 11.)

**One Grounded Sheath $w$.** With sheath $w$ of cable $abc$ grounded, and conductors $a$, $b$, $c$ connected in parallel, assume a voltage $E$ applied between ground and one terminal of the conductors with the other terminal grounded. The voltage drop around the closed loop consisting of conductor $a$ with ground return is

$$E = I_{a0}(Z_{aa-g} + Z_{ab-g} + Z_{ac-g}) + I_w Z_{aw-g} \qquad [38]$$

where $I_{a0}$ is the current in conductors $a$, $b$, and $c$, and $I_w$ is the current in sheath $w$ measured in the direction of $I_{a0}$.

The voltage drop in sheath $w$ with earth return is

$$I_w Z_{ww-g} + 3I_{a0} Z_{aw-g} = 0$$

$$I_w = -3I_{a0}\frac{Z_{aw-g}}{Z_{ww-g}} = -3I_{a0}\frac{Z_{aw-g}}{r_w + Z_{aw-g}} \qquad [39]$$

In a three-conductor cable, $Z_{ab-g} = Z_{ac-g}$. If $I_w$ in [38] is replaced by its value from [39], and [38] then divided by $I_{a0}$, $Z_{00-w}$ (the zero-sequence self-impedance with grounded sheath $w$) is obtained.

$$Z_{00-w} = \frac{E}{I_{a0}} = Z_{aa-g} + 2Z_{ab-g} - 3\frac{Z^2_{aw-g}}{r_w + Z_{aw-g}}$$

$$= (Z_{aa-g} + 2Z_{ab-g} - 3Z_{aw-g}) + 3\frac{r_w Z_{aw-g}}{r_w + Z_{aw-g}} \qquad [40]$$

The term in parentheses in [40] represents the internal component of zero-sequence self-impedance. In this term, three times the impedance of the earth-return path $Z_{aw-g}$ common to sheath $w$ and conductors $a$, $b$, and $c$ is subtracted from $Z_{aa-g} + 2Z_{ab-g}$, leaving only their internal components. The second term in [40] represents the external component of zero-sequence self-impedance when there is only one grounded sheath $w$.

Zero-sequence equations will be expressed in general form before the case of one grounded sheath is discussed further.

**Internal and External Components of Zero-Sequence Self-Impedance.** Let the zero-sequence self-impedance of circuit $abc$ with grounded sheath $w$ and any number of other grounded sheaths be expressed as the sum of its internal and external components $Z'_{00}$ and $Z_e$, respectively.

$$Z_{00} = Z'_{00} + Z_e = Z'_{00} + 3\frac{r_w Z_{ex}}{r_w + Z_{ex}} \qquad [41]$$

where

$$Z'_{00} = (Z_{aa-g} + 2Z_{ab-g} - 3Z_{aw-g}) \qquad [42]$$

$$Z_e = 3\frac{r_w Z_{ex}}{r_w + Z_{ex}} \qquad [43]$$

The impedance $Z_{ex}$ in [43] is the component of impedance in parallel with $r_w$, the resistance of the sheath $w$ of the cable $abc$ whose zero-sequence self-impedance is to be determined. The factor 3 in [43] expresses $Z_e$ on a per phase basis. $Z_{ex}$ depends upon the number of grounded sheaths, their dimensions and location, earth resistivity, and extraneous conducting elements. It is to be evaluated for a given system.

**Internal Component of Zero-Sequence Self-Impedance.** $Z'_{00}$ is constant at a given temperature for a cable with *non-magnetic* binder and may be evaluated by substituting [28], [29], and [34] in [42]:

$$Z'_{00} = r_a + j\left(0.159\frac{f}{60}\log_{10}\frac{\frac{d_w}{2}}{\sqrt[3]{\left(\frac{d_a}{2}\right)s_{ab}^2}} + x_i\right) \text{ohms per 1000 feet} \qquad [44]$$

where $r_a$ and $x_i$ are effective a-c resistance and internal reactance of one conductor in ohms per 1000 feet; $d_a$ and $d_w$ are outside diameters of conductor $a$ and sheath $w$; and the total reactance term in parentheses corresponds to the flux within the periphery of the sheath produced by zero-sequence currents in conductors $a$, $b$, and $c$.

The internal zero-sequence self-impedance $Z'_{00}$ for a cable with non-magnetic binder can be calculated from either [42] or [44]. If [42] is used, $X_{aa-g}$, $X_{ab-g}$, and $X_{aw-g}(= X_{ww-g})$ can be read from Figs. 8 to 10 and expressed in ohms per 1000 feet. As [42] requires the subtraction of relatively large quantities of comparable magnitudes, a higher degree of precision will usually be obtained by using [44].

**Cables with Magnetic Binders.** If there is a magnetic binder, correcting factors can be applied to the resistance and reactance components of $Z'_{00}$ calculated from [44].

The internal zero-sequence self-impedance of a 60-cycle 27-kv three-conductor shielded cable with sector conductors of 350,000 cir mils, 350-mil solid paper insulation, 4-mil steel binding tape, and outside sheath diameter of approximately 3 inches was determined by laboratory tests[12,13] at various current densities. (Results recorded in Fig. 4 of reference 12 and Fig. 12 of reference 13.) As the zero-sequence current in the conductors was varied from a very low value to a value ($3I_{a0} = 545$ amp, approx.) which saturated the binder, the resistance and reactance components of internal zero-sequence impedance increased to their maximum values of approximately twice their corresponding values at very low current density and then decreased to their constant saturated values. At saturation, the zero-sequence internal-reactance component was approximately 35% higher than its value at very low current density; the resistance component was not appreciably different for its value at very low current density.

The results of these tests can be applied approximately to other three-conductor cables with magnetic binders if the total zero-sequence current $3I_{a0}$ is high enough to saturate the binder (600 amp, approx.). This is usually the case when line-to-ground faults occur on grounded cable systems. Under this condition, until further tests are available, it is suggested that correction to the resistance component of $Z'_{00}$ be neglected and the reactance component given by [44] be increased by 35%. As the internal impedance is only part of the total impedance, errors in the internal resistance and reactance components do not produce proportionate errors in the total zero-sequence self-impedance.

**Zero-Sequence Currents in Grounded Sheaths and Ground.** The current $I_w$ in the sheath $w$ meets the resistance $r_w$; the balance of the return current $-(3I_{a0} + I_w)$ meets the external impedance $Z_{ex}$, which is in parallel with $r_w$. The current $I_w$ in the sheath $w$ and $I_{ex}$ in $Z_{ex}$, in the direction of $I_{a0}$, are

$$I_w = -3I_{a0}\frac{Z_{ex}}{r_w + Z_{ex}} \qquad [45]$$

$$\begin{aligned} I_{ex} &= -3I_{a0}\frac{r_w}{r_w + Z_{ex}} \\ &= -(3I_{a0} + I_w) = I_g + I_v + I_u + \cdots + I_n \qquad [46] \end{aligned}$$

$I_{ex}$ includes currents in the ground, in grounded sheaths exclusive of $w$, and in miscellaneous conducting elements.

**Zero-Sequence Mutual Impedance between Parallel Three-Phase Cable Circuits.** The voltage drop $V_{A0}$ induced in the phase conductors of cable *ABC* with sheath *v*, in the direction of $I_{a0}$, is

$$V_{A0} = 3I_{a0}Z_{Aa-g} + I_wZ_{Aw-g} + I_vZ_{Av-g} + I_uZ_{Au-g} + \cdots + I_nZ_{An-g} \qquad [47]$$

If $Z_{Aa-g}$ and $Z_{Aw-g}$ in [47] are replaced by $Z_{wv-g}$, $Z_{Au-g}$ by $Z_{vu-g}$, $Z_{An-g}$ by $Z_{vn-g}$, $I_w$ by its values given in [45], and [47] then divided by $I_{a0}$, the mutual impedance $Z_{0m}$ between circuits *abc* and *ABC* is obtained:

$$Z_{0m} = \frac{V_{A0}}{I_{a0}} = \frac{3r_wZ_{wv-g}}{r_w + Z_{ex}} + \frac{I_v}{I_{a0}} Z_{Av-g} + \frac{I_u}{I_{a0}} Z_{vu-g} + \cdots + \frac{I_n}{I_{a0}} Z_{vn-g} \qquad [48]$$

**Evaluation of $Z_{ex}$.** The external impedance $Z_e$ can be determined from [43] after the impedance $Z_{ex}$ in parallel with $r_w$ has been evaluated. One method of evaluating $Z_{ex}$ is to derive first the equation for $I_w$, the current in the sheath of the faulted cable, and then equate this value of $I_w$ to $I_w$ in [45] and solve for $Z_{ex}$. Equations have been developed in Chapter XI, Volume I, for overhead transmission circuits with one, two, and three grounded wires. From these equations, $Z_{ex}$ can be evaluated for one, two, and three grounded sheaths or groups of sheaths.

An alternate method of evaluating $Z_{ex}$, and deriving equations for $Z_{00}$, $Z_{0m}$, $I_w$, $I_v$, $I_u$, $\cdots$, $I_g$ is by means of equivalent circuits. Both methods will be given. The procedure in determining $Z_{ex}$ by means of field test will be outlined later.

**One Grounded Sheath *w* (Continued).** Equating $I_w$ in [45] to $I_w$ in [39] of this chapter, and solving for $Z_{ex}$,

$$Z_{ex} = Z_{aw-g} = Z_{ww-g} - r_w \qquad [49]$$

With $Z_{ex}$ known, $Z_{00-w}$, $I_w$, and $I_g = I_{ex}$ are calculated from [41], [45], and [46], respectively. $I_w$ and $Z_{00-w}$ (previously derived) are given by [39] and [40]. As there is only one grounded sheath, $I_v = I_u = I_n = 0$. From [46],

$$I_g = I_{ex} = -3I_{a0}\frac{r_w}{r_w + Z_{aw-g}} \qquad [50]$$

From [48], the mutual impedance between circuit *abc* with grounded sheath *w* and any multiconductor cable with *ungrounded* sheath *v*, is

$$Z_{0m-w} = \frac{3r_wZ_{wv-g}}{r_w + Z_{aw-g}} \qquad [51]$$

**Two Grounded Sheaths *w* and *v*.** (See Fig. 11.) Equations [61] and [62] of Chapter XI, Volume I, give $I_w$ and $I_v$ in terms of $I_{a0}$. If $I_w$ in [61] is equated to $I_w$ in [45], $Z_{ex}$ can be evaluated. The following equations give $Z_{ex}$ and $I_v$:

$$Z_{ex} = Z_{aw-g} - \frac{Z^2_{wv-g}}{Z_{vv-g}} \qquad [52]$$

$$I_v = -3I_{a0}\frac{r_w}{r_w + Z_{ex}}\left(\frac{Z_{wv-g}}{Z_{vv-g}}\right) \qquad [53]$$

$Z_{00-wv}$, $I_w$, and $I_{ex} = I_v + I_g$ are obtained by substitution of $Z_{ex}$ from [52] in [41], [45], and [46], respectively.

The zero-sequence self-impedance $Z_{00-wv}$ of the circuit *abc* is

$$Z_{00-wv} = Z'_{00} + \frac{3r_w Z_{ex}}{r_w + Z_{ex}} = Z'_{00} + \frac{3r_w\left(Z_{aw-g} - \dfrac{Z^2_{wv-g}}{Z_{vv-g}}\right)}{r_w + \left(Z_{aw-g} - \dfrac{Z^2_{wv-g}}{Z_{vv-g}}\right)} \qquad [54]$$

$$I_g = I_{ex} - I_v = -3I_{a0}\frac{r_w}{r_w + Z_{ex}}\left(\frac{Z_{vv-g} - Z_{wv-g}}{Z_{vv-g}}\right) \qquad [55]$$

$I_w$, $I_v$, and $I_g$ are not in phase and therefore their vector sum $-3I_{a0}$ is not as high as their scalar sum.

The zero-sequence mutual impedance $Z_{0m-wv}$ between circuits *abc* and *ABC* with sheaths *w* and *v* grounded, obtained by substitution of $I_v/I_{a0}$ from [53] in [48] with $I_u = I_n = 0$, and $Z_{Av-g}$ replaced by $Z_{vv-g} - r_v$, is

$$Z_{0m-wv} = \frac{3r_w r_v \dfrac{Z_{wv-g}}{Z_{vv-g}}}{r_w + Z_{ex}} = -r_v\frac{I_v}{I_{a0}} \qquad [56]$$

When sheaths *w* and *v* are identical,

$$Z_{ww-g} = Z_{vv-g} = r_w + R_g + jX_{ww-g} = r_w + Z_{aw-g} = r_v + Z_{Av-g}$$

$$Z_{aw-g} = Z_{Av-g} = R_g + jX_{ww-g} = Z_{ww-g} - r_w$$

**Sheath *w* and Several Additional Identical Grounded Sheaths.** The equations developed above with sheath *w* of cable *abc* grounded, and one additional grounded sheath *v*, are approximately applicable when there are several additional grounded sheaths *x*, *y*, *z*, etc., if $Z_{vv-g}$ represents the equivalent self-impedance with earth return of sheaths *x*, *y*, *z*, etc., and $Z_{av-g} = Z_{wv-g}$ is the average mutual impedance with common earth return between phase conductors *a*, *b*, *c* (or sheath $w$) and sheaths $x, y, z$, etc. This average mutual impedance corresponds to the geometric mean distance between the center of sheath $w$ and the centers of sheaths $x, y, z$, etc. The equivalent self-impedance $Z_{vv-g}$ is determined as follows:

For two identical sheaths $x$ and $y$:

$$Z_{vv-g} = \tfrac{1}{2}(Z_{xx-g} + Z_{xy-g})$$

For three identical sheaths $x, y, z$:

$$Z_{vv-g} = \tfrac{1}{3}(Z_{xx-g} + 2Z_{xy-g}) \qquad [57]$$

For $n$ identical sheaths:

$$Z_{vv-g} = \frac{1}{n}\,[Z_{xx-g} + (n-1)Z_{xy-g}]$$

where $n$ is the number of sheaths in addition to sheath $w$; $Z_{xx-g}$ is the self-impedance with earth return of any one of the identical sheaths $x, y, z$, etc., and $Z_{xy-g}$ is the mutual impedance with common earth return corresponding to the geometric mean spacing between the sheaths $x, y, z$, etc.

Replacing two or more identical grounded sheaths by one equivalent grounded sheath is based on the assumption that the sheaths, exclusive of the sheath $w$ of the faulted cable *abc* whose zero-sequence self-impedance is to be determined, carry equal currents. This is the case only when they are equidistant from sheath $w$ and, when there are more than two sheaths, equidistant from each other as well. However, the use of one equivalent sheath to replace several other identical sheaths produces but slight error in calculated zero-sequence impedances.

**Sheath *w* and Two Additional Dissimilar Grounded Sheaths *v* and *u*.** If the sheaths exclusive of sheath $w$ are dissimilar, or if there are sheaths and ground wires, it is necessary to consider two additional sheaths $v$ and $u$, or an equivalent sheath and a ground wire or its equivalent. In Volume I, page 399, equations are given for the currents $I_w$, $I_v$, and $I_u$ in three ground wires $w$, $v$, and $u$. These equations are applicable to sheath $w$ of cable *abc*, sheath $v$ of cable *ABC*, and to an additional sheath or group of similar grounded sheaths $u$. If numerators and denominators of these equations are divided by $(Z_{vv-g}Z_{uu-g} - Z^2_{vu-g})$ and $I_w$ equated to $I_w$ in [45], the impedance $Z_{ex}$, and $I_v$ and $I_u$ in terms of $I_{a0}$, can be determined.

$$Z_{ex} = \left[Z_{aw-g} - \frac{Z_{vv-g}Z^2_{wu-g} + Z_{uu-g}Z^2_{wv-g} - 2Z_{wv-g}Z_{wu-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{vu-g}}\right] \qquad [58]$$

$$I_v = -3I_{a0}\frac{r_w}{r_w + Z_{ex}}\,\frac{Z_{wv-g}Z_{uu-g} - Z_{wu-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{vu-g}} \qquad [59]$$

$$I_u = -3I_{a0}\frac{r_w}{r_w + Z_{ex}}\,\frac{Z_{wu-g}Z_{vv-g} - Z_{wv-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{vu-g}} \qquad [60]$$

$I_w$, $I_{ex}$, and $Z_{00-wvu}$ are obtained by substitution of $Z_{ex}$ in [45], [41], and [46], respectively.

$$I_g = I_{ex} - I_v - I_w \qquad [61]$$

By replacing $I_v$ and $I_u$ in [48] by their values given by [59] and [60] and simplifying, the zero-sequence mutual impedance between circuit *abc* and circuit *ABC* with grounded sheaths *w*, *v*, and *u*, is

$$Z_{0m-wvu} = 3\frac{r_w r_v}{r_w + Z_{ex}}\left[\frac{Z_{uu-g}Z_{wv-g} - Z_{wu-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{vu-g}}\right] = -r_v\frac{I_v}{I_{a0}} \qquad [62]$$

**Equivalent Circuit for Determining Zero-Sequence Impedances.** The equations given above for three-conductor cables were evaluated from work previously done on overhead transmission lines with ground wires. These equations can also be derived from the equivalent circuits of Figs. 12 to 14.

Figure 12(*a*) shows two three-conductor cables *abc* and *ABC* with sheaths *w* and *v*, respectively. Figure 12(*b*) gives the zero-sequence equivalent circuit constructed on a per phase basis: all elements in the return path are multiplied by three and all return currents divided by three. Arrows accompanying $I_w/3$, $I_v/3$, and $I_g/3$ indicate the direction of current flow — not that assumed as positive in deriving the equations. $Z'_{00}(a)$ is the zero-sequence internal impedance of cable *abc*, $Z'_{00}(A)$ is that of cable *ABC*.

A voltage $E$ is applied between circuit *abc* and ground with the connection at $A$ open. $I_{a0}$ is the per phase zero-sequence current. $I_w$, $I_v$, and $I_g$ are three times the currents at *w*, *v*, and *g*, respectively. The zero-sequence self-impedance of the cable *abc* is $E/I_{a0}$. The voltage to ground $V_A$ at $A$ is the voltage induced in circuit *ABC*. The zero-sequence mutual impedance between circuits *abc* and *ABC* is $V_A/I_{a0}$. If sheath *w* (or *v*) is open-circuited, the connection to ground is broken at *w* (or *v*). If sheaths *w* and *v* are both open-circuited, all the current returns in the ground: $I_g/3 = -I_{a0}$, and the external impedance $Z_e = 3Z_{aw-g}$. If sheath *w* is grounded and *v* open-circuited, equations [39], [40], and [49]–[51] are obtained. If both sheaths are grounded, equations [52]–[56] are obtained. The zero-sequence self-impedance of circuit *ABC* and the currents $I_w$, $I_v$, and $I_g$ are determined in a similar manner by applying a voltage at $A$ between circuit $ABC$ and ground with the connection at $a$ open.

Figure 13($a$) shows three three-conductor cables with sheaths $w$, $v$, and $u$. Cable $A'B'C'$ with sheath $u$ is equidistant from the other two. Figure 13($b$), similar to Fig. 12($b$), is the zero-sequence equivalent

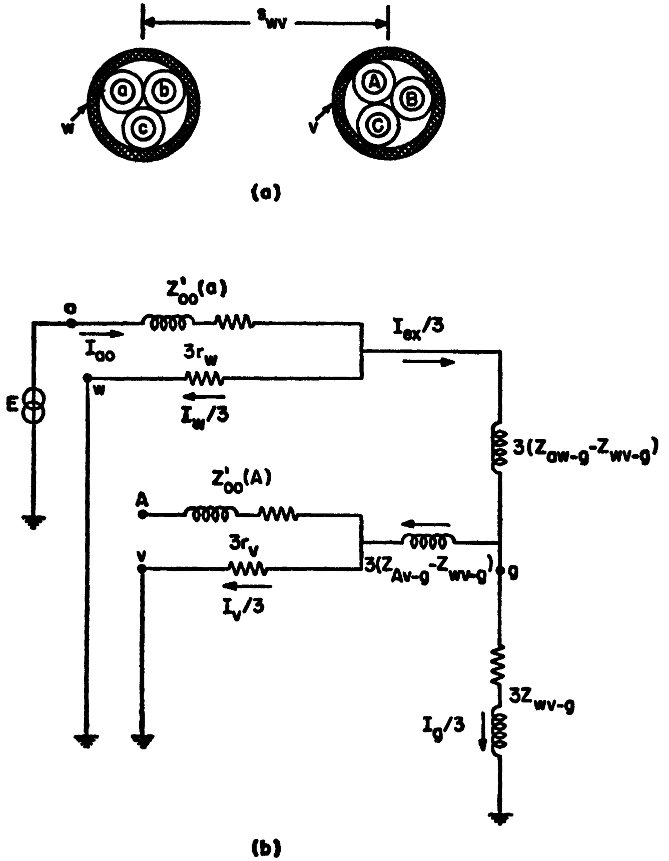

FIG. 12. ($a$) Two three-conductor cables with grounded sheaths. ($b$) Zero-sequence equivalent circuit for ($a$) with voltage to ground $E$ applied at $a$.

circuit of Fig. 13($a$). The zero-sequence self-impedance per phase of any one of the three-phase circuits $abc$, $ABC$, or $A'B'C'$ and the mutual impedances between it and the other two circuits can be obtained by applying a voltage to ground $E$ at terminal $a$, $A$, or $A'$, respectively, with the other two terminals open. Figure 13($b$) can be used to derive equations [58]–[62] when $s_{wu} = s_{vu}$, and therefore $Z_{wu-g} =$

$Z_{vu-g}$. With $s_{wu} = s_{vu}$, the equivalent circuit of Fig. 13(b) is constructed without mutual coupling devices. If distances between the three cables are unequal, or if there are more than three sheaths to be considered, the a-c network analyzer can be used to advantage.

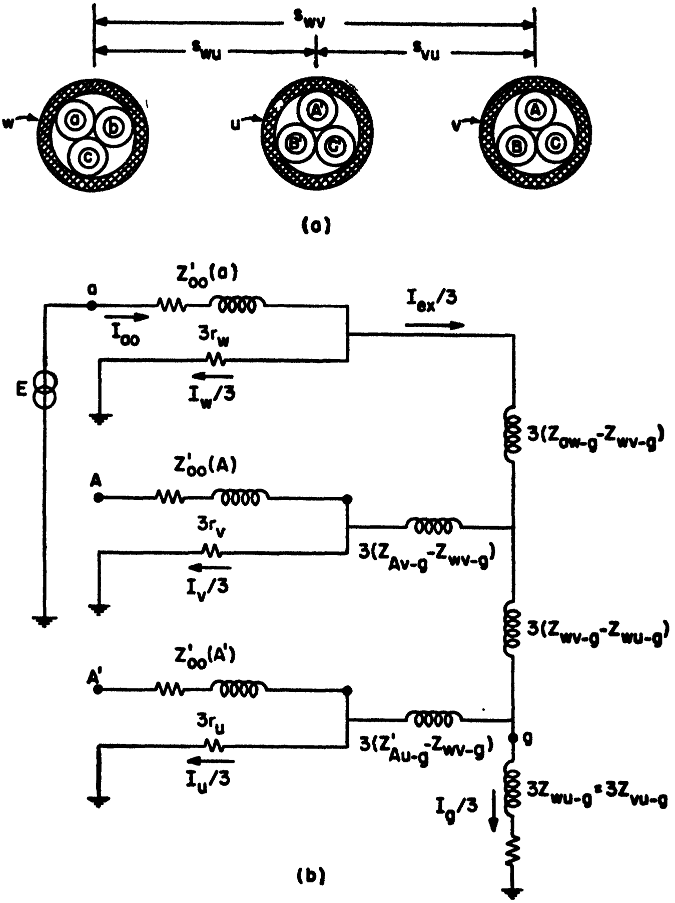

FIG. 13. (a) Three three-conductor cables with grounded sheaths, where one cable is equidistant from the other two. (b) Zero-sequence equivalent circuit for (a).

**A-C Network Analyzer Equivalent Circuits.** Figure 14(a) shows three three-conductor cables and one additional grounded sheath, ground wire, or pipe indicated by *p*. Figure 14(b), which is a zero-sequence equivalent circuit of Fig. 14(a) for use on the a-c network analyzer, is similar to Figs. 12(b) and 13(b) except for the mutual coupling transformers with 1 : 1 turn ratios. It is used in the same way to determine the zero-sequence self-impedance of any one of the three-phase circuits, and the zero-sequence mutual impedance between these circuits. Currents in sheaths and ground, with zero-sequence current in any circuit, are also similarly determined.

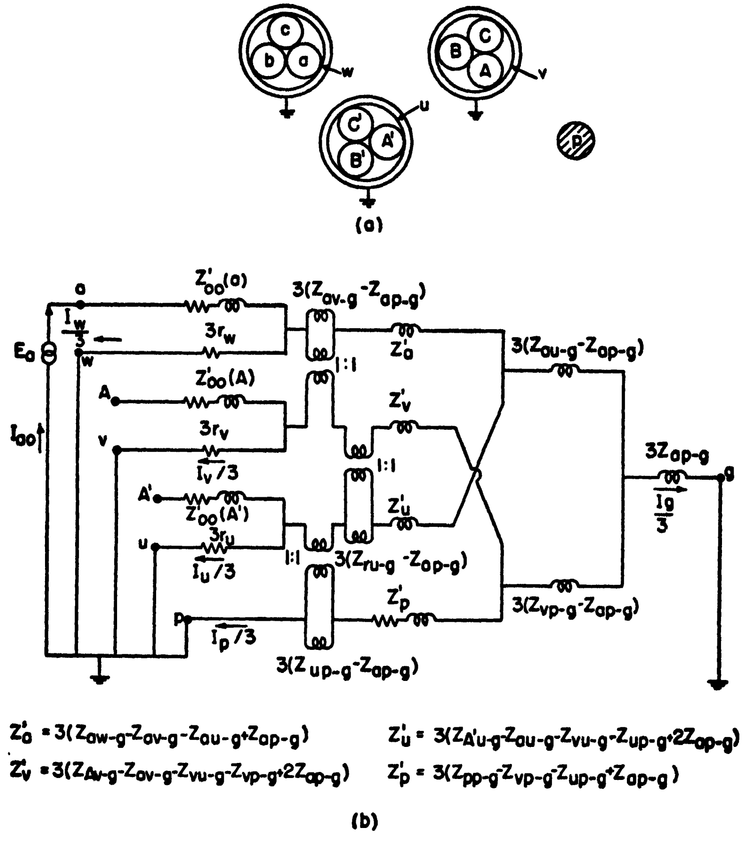
c
b
a
w
C
B
A
v
u
C'
A'
B'
P
(a)
a
$Z'_{oo}(a)$
$3(Z_{av\text{-}g}-Z_{ap\text{-}g})$
$I_w/3$
$3r_w$
$E_o$
w
$Z'_a$
1:1
$Z'_{oo}(A)$
A
$3(Z_{au\text{-}g}-Z_{ap\text{-}g})$
$Z'_v$
$I_{oo}$
$3r_v$
v
$I_v/3$
1:1
$3Z_{ap\text{-}g}$
g
$I_g/3$
A'
$Z'_{oo}(A')$
$Z'_u$
$3r_u$
u
$I_u/3$
1:1
$3(Z_{ru\text{-}g}-Z_{ap\text{-}g})$
$Z'_p$
$3(Z_{vp\text{-}g}-Z_{ap\text{-}g})$
p
$I_p/3$
$3(Z_{up\text{-}g}-Z_{ap\text{-}g})$
$Z'_a = 3(Z_{aw\text{-}g}-Z_{av\text{-}g}-Z_{au\text{-}g}+Z_{ap\text{-}g})$
$Z'_u = 3(Z_{A'u\text{-}g}-Z_{au\text{-}g}-Z_{vu\text{-}g}-Z_{up\text{-}g}+2Z_{ap\text{-}g})$
$Z'_v = 3(Z_{Av\text{-}g}-Z_{av\text{-}g}-Z_{vu\text{-}g}-Z_{vp\text{-}g}+2Z_{ap\text{-}g})$
$Z'_p = 3(Z_{pp\text{-}g}-Z_{vp\text{-}g}-Z_{up\text{-}g}+Z_{ap\text{-}g})$
(b)

FIG. 14. (*a*) Three three-conductor cables with grounded sheaths and one pipe, ground wire, or additional sheath *P*. (*b*) Zero-sequence equivalent circuit for (*a*) for use on an a-c network analyzer.

The equivalent circuit of Fig. 14(*b*) can be extended to include any number of parallel cables with grounded sheaths, and any number of ground wires or pipes. Such equivalent circuits are limited only by the number of mutual coupling transformers available on the a-c network analyzer. For pipes which parallel cables, the required information is their diameters and internal inductances. Their self-impedances with earth return, and the mutual impedances with common earth return between pipes and sheaths or ground wires, can be obtained from [28]–[32]. Where two or more pipes can be replaced by a single equivalent pipe, the number of circuit elements is reduced. As previously stated, when the nature of miscellaneous conducting material extraneous to the cable system is unknown, accurate values of zero-sequence self- and mutual impedances can be obtained only by field tests.

**Determination of the Impedance $Z_{ex}$ in Parallel with $r_w$, the Sheath Resistance of the Cable abc, by Means of Field Tests.** $E_{ex}$, at a given location in a given system, can be determined by tests as follows: The zero-sequence self-impedance of circuit *abc* is determined by test as the ratio of the zero-sequence voltage applied to the circuit to the average current in the phases.

$Z'_{00}$, the internal zero-sequence impedance of circuit *abc* with sheath *w* is calculated from [44] for a three-conductor cable with non-magnetic binder; it may be determined by test for cables with magnetic binders, as described in reference 12 or 13. The resistance $r_w$ of sheath *w* is determined by calculation or test. $Z_{ex}$, the external impedance in parallel with $r_w$, is then evaluated from [41] where $Z_{00}$ represents the measured zero-sequence self-impedance of circuit *abc*. $Z_{ex}$, determined in this way, includes all grounded sheaths (exclusive of sheath *w*), ground wires, ground, and miscellaneous conducting material. The general effect of extraneous conducting material, which parallels cable circuits, is to increase the resistance component and lower the reactance component of $Z_{ex}$ calculated with such conducting material neglected.

Example 4 illustrates the calculation of zero-sequence impedances of three-conductor cables with grounded lead sheaths. In these calculations, the resistance of the conductors is corrected for temperature and skin effect. No correction is made for proximity effect. (See discussion of zero-sequence proximity effect in Chapter II.)

**Example 4.** Calculate the zero-sequence self- and mutual impedances, and the sheath and ground currents in terms of $I_{a0}$, of the 60-cycle three-conductor cables with grounded sheaths shown in Fig. 15. Distance between duct centers is 4 inches. As there are tap points between buses, the self-impedance of each circuit and the mutual impedance between circuits are required. All cables are identical belted cables with 300,000-cm non-compact sector copper conductors. Thickness of conductor insulation, belt, and lead sheath are $\frac{13}{64}$ inch, $\frac{13}{64}$ inch, and $\frac{8}{64}$ inch, respectively. It is estimated that conductor and sheath temperatures are 70°C and 40°C, respectively; and that the earth resistivity $\rho$ is 100 ohms (meter cube). There are no cables with grounded sheaths other than those indicated in Fig. 15; and there are no known miscellaneous conducting elements.

As a matter of interest, determine the zero-sequence self- and mutual impedances which would result if all zero-sequence current returned in the ground, and also if all returned in the sheath of the cable whose conductors carry zero-sequence current.

*Solution.* Let the three-conductor cable whose zero-sequence self-impedance is to be determined be *abc* with sheath *w*. In Fig. 15, there are three cases to be considered: (*a*) one, (*b*) two, and (*c*) four cables with grounded sheaths in addition to the cable *abc* with grounded sheath *w*.

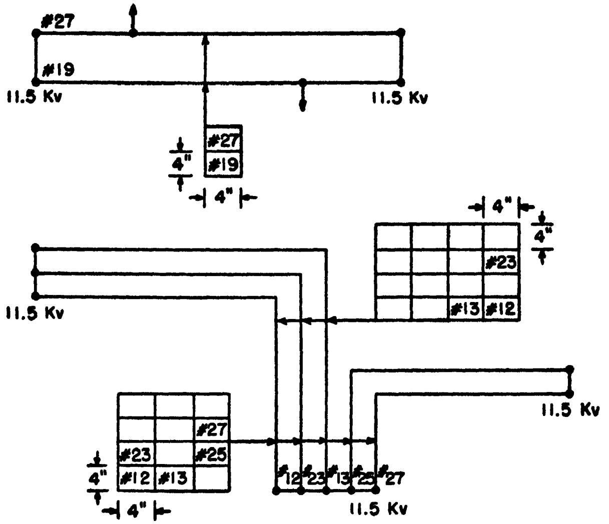

FIG. 15. Duct line for Example 4. All cables are 300,000 cir mils sector three-conductor (copper), belted cables with $\frac{13}{64}$-in. belt, $\frac{13}{64}$-in. conductor insulation, and $\frac{8}{64}$-in. lead thickness.

(*a*) *Two Identical Cables in Duct Bank.* Let the second cable be *ABC* with sheath *v*. Equations [52]–[56] apply to this case.

As thicknesses of conductor insulation, belt, and lead are given, the factor 1.03 will be omitted in the calculation of sheath diameters. From [10] and [11], $D_i$ and $D_o$ are

$$D_i = 2\sqrt{0.300} + 4.5\left(\tfrac{13}{64}\right) + 2\left(\tfrac{13}{64}\right) = 2.417 \text{ inches}$$

$$D_o = D_i + 2\left(\tfrac{8}{64}\right) = 2.417 + 0.250 = 2.667 \text{ inches}$$

From [12] or Fig. 3, $r_w$ = 0.113 ohm per 1000 feet = sheath resistance. From Table III, Appendix A,

$d_a$ = 0.631 inch = diameter of round conductor of same circular mils

$\dfrac{R'}{R_{dc}}$ = 1.006 = approximate skin-effect resistance factor

$r_a$ = 0.0423 (at 70°C) × 1.006 = 0.0426 ohm per 1000 feet = conductor resistance, corrected for temperature and skin effect

The equivalent spacing $s_{ab} = s$ between sector conductors, evaluated from [9] by replacing $D_t$ by 2.417, $2t$ by $2(\frac{13}{64})$, omitting the factor 1.03, and solving for $s$, is

$$s = s_{ab} = \frac{2.417 - 0.406}{2.155} = 0.934 \text{ inch}$$

Determined from [7] with $d$ replaced by $0.836d$, $s = s_{ab} = 0.836 \times 0.631 + 0.406 = 0.934$ inch. From Table II, Chapter II, $(x_i)_a = 0.00575$ ohm per 1000 feet (approximately that of a solid round conductor). From [28]–[35] and Fig. 8, in ohms per 1000 feet

$$Z_{aa-g} = (0.0426 + 0.018) + j\left(\frac{1.405}{5.28} + 0.006\right) = 0.0606 + j0.272$$

$$Z_{ab-g} = 0.018 + j\frac{1.27}{5.28} = 0.018 + j0.241$$

$$Z_{ww-g} = Z_{vv-g} = (0.113 + 0.018) + j\frac{1.23}{5.28} = 0.131 + j0.233 = 0.267\underline{/60.6^\circ}$$

$$Z_{aw-g} = Z_{AV-g} = Z_{ww-g} - r_w = 0.018 + j0.233$$

From Fig. 15, $s_{wv} = s_{Aa} = 4$ inches.

$$Z_{wv-g} = Z_{aA-g} = 0.018 + j\frac{1.095}{5.28} = 0.018 + j0.208 = 0.209\underline{/85.0^\circ}$$

The internal zero-sequence self-impedance from [42] is

$$\begin{aligned} Z'_{00} &= 0.0606 + j0.272 + 2\ (0.018 + j0.241) - 3\ (0.018 + j0.233) \\ &= 0.0426 + j0.055 \end{aligned}$$

Calculated from [44], for greater precision and for check,

$$Z'_{00} = 0.4426 + j\left[0.159 \log_{10} \frac{\frac{2.667}{2}}{\sqrt[3]{\frac{0.631}{2}\ (0.934)^2}} + 0.0058\right] = 0.0426 + j0.0553$$

From [52],

$$\begin{aligned} Z_{ex} &= 0.018 + j0.233 - \frac{(0.209\underline{/85.0^\circ})^2}{0.267\underline{/60.6^\circ}} \\ &= 0.018 + j0.233 - (-0.054 + j0.154) = 0.072 + j0.079 = 0.107\underline{/47.7^\circ} \end{aligned}$$

$$r_w + Z_{ex} = 0.185 + j0.079 = 0.201\underline{/23.1^\circ}$$

$$\frac{3r_w Z_{ex}}{r_w + Z_{ex}} = 0.1805\underline{/24.6^\circ} = 0.164 + j0.075$$

From [54], the zero-sequence self-impedance $Z_{00-wv}$ of either cable on a per phase basis in ohms per 1000 feet is

$$Z_{00-wv} = (0.0426 + j0.0553) + (0.164 + j0.075) = 0.207 + j0.130$$

From [45] and [53],

$$I_w = -3I_{a0}\frac{0.107\angle 47.7^\circ}{0.201\angle 23.1^\circ} = -3I_{a0}\,(0.532\angle 24.6^\circ) = -3I_{a0}\,(0.483 + j0.221)$$

$$I_v = -3I_{a0}\frac{0.113}{0.201\angle 23.1^\circ}\times\frac{0.209\angle 85.0^\circ}{0.267\angle 60.6^\circ} = -I_{a0}\,(0.439\angle 1.3^\circ)$$

$$= -3I_{a0}(0.439 + j0.010)$$

$$I_g = -(3I_{a0} + I_w + I_v) = -3I_{a0}\,(0.078 - j0.231) = -3I_{a0}\,(0.243\angle -71.3^\circ)$$

From [56], the zero-sequence mutual impedance $Z_{0w-wv}$ between the two circuits is

$$Z_{0m-wv} = -r_v\frac{I_v}{I_0} = 0.113\times 3(0.439 + j0.010) = 0.149 + j0.003 \text{ ohm per 1000 feet}$$

(*b*) *Three Identical Cables in Duct Bank.* From Fig. 15, the three distances between cable axes are unequal. In the general case of three or more cables in the duct bank, the zero-sequence self-impedances of the cables will differ slightly from each other, and the mutual impedance between any two cables will differ from that between any other two. The *average* zero-sequence self- and mutual impedances can be determined by replacing all cable except *abc* with sheath *w* by an equivalent cable *ABC* with an equivalent sheath *v*, where $Z_{vv-g}$ is defined in [57], and the mutual impedances with common earth return $Z_{aA-g} = Z_{wv-g}$ corresponds to the geometric mean distance (*gmd*) between axes of sheaths. If *x* and *y* indicate the sheaths of the two additional cables,

$$Z_{wv-g} = Z_{wx-g} = Z_{wy-g} = Z_{xy-g}$$

where this average mutual impedance with common earth return from Fig. 15 corresponds to $s_{wv} = \sqrt[3]{(4)(8)\sqrt{4^2 + 8^2}} = 6.6$ inches.

In ohms per 1000 feet,

$$Z_{wv-g} = 0.018 + j\frac{1.03}{5.28} = 0.018 + j0.195 = 0.196\angle 84.7^\circ$$

From part (*a*), $r_w = 0.113$; $Z_{ww-g} = 0.131 + j0.233$; $Z_{aw-g} = 0.018 + j0.233$; $Z'_{00} = 0.0426 + j0.0553$. From [57], with two sheaths in addition to sheath *w*,

$$Z_{vv-g} = \tfrac{1}{2}\,(Z_{xx-g} + Z_{xy-g}) = \tfrac{1}{2}\,(Z_{ww-g} + Z_{wv-g})$$

$$= \tfrac{1}{2}\,[(0.131 + j0.233) + (0.018 + j0.195)] = 0.0745 + j0.214 = 0.227\angle 70.8^\circ$$

$$r_v = \tfrac{1}{2}\,r_w = 0.0565$$

$$Z_{Av-g} = Z_{vv-g} - r_v = 0.018 + j0.214 = 0.215\angle 85.1^\circ$$

= mutual impedance with earth return between any conductor *A* and sheath *v* of the equivalent cable which replaces the two additional cables

With $Z_{ww-g}$, $Z_{aw-g}$, $Z_{vv-g}$, and $Z_{Av-g}$ known, equations [52]–[56] can be applied

$$\frac{Z^2_{wv-g}}{Z_{vv-g}} = \frac{(0.196\angle 84.7^\circ)^2}{0.227\angle 70.8^\circ} = 0.169\angle 98.6^\circ = -0.025 + j0.167$$

From [52],

$$Z_{ex} = 0.018 + j0.233 - (-0.025 + j0.167) = 0.043 + j0.066 = 0.079\underline{/56.9^\circ}$$

$$r_w + Z_{ex} = 0.156 + j0.066 = 0.169\underline{/22.9^\circ}$$

$$\frac{3r_wZ_{ex}}{r_w + Z_{ex}} = \frac{3 \times 0.113 \times 0.079\underline{/56.9^\circ}}{0.169\underline{/22.9^\circ}} = 0.158\underline{/34.0^\circ} = 0.131 + j0.089$$

The average zero-sequence self-impedance of the three cables from [54] is

$$Z_{00-wv} = Z_{00-wxy} = (0.043 + j0.055) + (0.131 + j0.089) = 0.174 + j0.144$$

From [45] and [53],

$$\begin{aligned} I_w &= -3I_{a0}(0.466\underline{/34.0^\circ}) &&= -3I_{a0}(0.386 + j0.260) \\ I_v &= -3I_{a0}(0.575\underline{/-9.0^\circ}) &&= -3I_{a0}(0.568 - j0.090) \\ I_g &= -(3I_{a0} + I_w + I_v) &&= -3I_{a0}(0.046 - j0.170) \\ I_x &= I_y = \tfrac{1}{2}\, I_v &&= -3I_{a0}(0.284 - j0.045) \end{aligned}$$

From [56], the average mutual impedance between cables is

$$Z_{0m-wv} = Z_{0m-wxy} = -r_v\frac{I_v}{I_{a0}} = 0.098\underline{/9.0^\circ} = 0.097 - j0.015$$

(*c*) *Five Identical Cables in Duct Bank.* This case is similar to part (*b*) and will be solved in a similar manner. With four cables in addition to cable *abc* with sheath *w*, $n = 4$ in [57] and the *gmd* between axes of cables is $\sqrt[10]{(4)^3(4\sqrt{2})^2(8)\,(4\sqrt{5})^3\,(8\sqrt{2})}$ = 6.5 inches. In ohms per 1000 feet,

$$Z_{wv-g} = 0.018 + j\frac{1.04}{5.28} = 0.018 + j0.197 = 0.198\underline{/84.8^\circ}$$

$Z_{ww-g}$, $Z_{aw-g}$, $r_w$, and $Z'_{00}$ same as in parts (*a*) and (*b*)

$$\begin{aligned} Z_{vv-g} &= \tfrac{1}{4}(Z_{ww-g} + 3Z_{wv-g}) = \tfrac{1}{4}[(0.131 + j0.233) + 3(0.018 + j0.197)] \\ &= 0.046 + j0.206 = 0.211\underline{/77.4^\circ} \end{aligned}$$

$$r_v = \frac{r_w}{4} = 0.028$$

$$Z_{Av-g} = Z_{vv-g} - r_v = 0.018 + j0.206$$

$$\frac{Z_{wv-g}}{Z_{vv-g}} = \frac{(0.198\underline{/84.8^\circ})^2}{0.211\underline{/77.4^\circ}} = 0.1855\underline{/92.2^\circ} = -0.007 + j0.185$$

$$Z_{ex} = 0.018 + j0.233 - (-0.007 + j0.185) = 0.025 + j0.048 = 0.0541\underline{/62.5^\circ}$$

$$r_w + Z_{ex} = 0.138 + j0.048 = 0.146\underline{/19.2^\circ}$$

$$\frac{3\,r_wZ_{ex}}{r_w + Z_{ex}} = 0.126\underline{/43.3^\circ} = 0.092 + j0.087$$

From [54], the average zero-sequence self-impedance of the five cables is $Z_{00-wv} = Z_{00-wuxyz} = (0.043 + j0.055) + (0.092 + j0.087) = 0.134 + j0.142$

From [45] and [53],

$$I_w = -3I_{a0}\frac{0.0541\angle 62.5^\circ}{0.146\angle 19.2^\circ} = -3I_{a0}(0.371\angle 43.3^\circ) = -3I_{a0}(0.270 + j0.254)$$

$$I_v = -3I_{a0}\frac{0.113}{0.146\angle 19.2^\circ} \times \frac{0.0198\angle 84.8^\circ}{0.211\angle 77.4^\circ} = 3I_{a0}(0.715\angle -11.8^\circ)$$

$$= -3I_{a0}(0.709 - j0.148)$$

$$I_g = -(3I_{a0} + I_w + I_v) = -3I_{a0}(0.021 - j0.106)$$

The average current in the four additional sheaths is

$$I_u = I_x = I_y = I_z = \frac{I_v}{4} = -3I_{a0}(0.181\angle -11.8^\circ) = -3I_{a0}(0.179 - j0.037)$$

From [56], the average zero-sequence mutual impedance between cables is

$$Z_{0m-wv} = Z_{0m-wuxyz} = -\frac{r_v I_v}{I_{a0}} = 0.0601 - j0.0125$$

$$= 0.0614\angle -11.8^\circ \text{ ohm per 1000 feet}$$

If all zero-sequence current returns in the ground (which could happen only with ungrounded sheaths), the zero-sequence self-impedance $Z_{00}$ of any one of the cables is

$$Z_{00} = Z_{aa-g} + 2Z_{ab-g} = Z'_{00} + 3Z_{aw-g} = 0.0966 + j0.754 \text{ ohms per 1000 feet}$$

The zero-sequence mutual impedance $Z_{0m}$ between circuits depends upon the distance between circuits. With $s_{aA} = 4$ inches,

$$Z_{0m} = 3Z_{aA-g} = 3(0.018 + j0.208) = 0.054 + j0.624 \text{ ohms per 1000 feet}$$

If all zero-sequence current returns in the sheath $w$ of cable $abc$ (which could happen only with infinite earth resistivity and the sheaths other than $w$ carrying no current),

$$Z_{00-w} = Z'_{00} + 3r_w = (0.0426 + j0.0553) + 3(0.113) = 0.382 + j0.055$$

$$I_w = -3I_{a0}$$

If all the current returns in the sheath $w$ of the cable carrying zero-sequence current, zero-sequence mutual impedances between cables will be zero. The voltage induced in a neighboring cable by the current in the three conductors will be completely canceled out by the voltage induced by the same current returning in the sheath.

## Circuits of Single-Conductor Cables

The sheaths of single-conductor cables may be operated short-circuited or open-circuited, as explained under " Methods of Sheath Operation " above. In determining zero-sequence impedances, there are two general cases to be considered: (1) no currents in the sheaths of the single-conductor cables and (2) currents in these sheaths.

**1. Sections of Cable Sheaths Insulated from Each Other with One Terminal Only Grounded, or Bonded to a Ground Wire or Grounded Neutral Conductor.** As the sheaths carry no longitudinal currents, they may be ignored in the calculation of zero-sequence impedances. With no miscellaneous conducting elements, calculations are similar to those for overhead circuits of the same number of phases and the same number of ground wires, all grounded sheaths in the duct bank being treated as ground wires.

Equations are given in Volume I, Chapter XI, pages 388–404, for determining zero-sequence self- and mutual impedances, and currents in ground wires and ground, in three-phase overhead transmission circuits with no ground wires, one ground wire, and two or three dissimilar ground wires. Equations for the zero-sequence self-impedances of single-phase and two-phase circuits with grounded neutral conductors are given in Volume I, page 428. These equations are applicable to three-phase and single-phase circuits of single-conductor cables, where no current flows in their sheaths, if all grounded sheaths in the duct bank are treated as ground wires.

*Voltages Induced in Conductors of a Multiconductor Control or Power Cable with Grounded Sheath.* Of interest is the voltage induced in any conductor $A$ of a multiconductor control or power cable with grounded sheath $v$, by zero-sequence currents in a three-phase circuit $abc$ of single-conductor cables with no currents in their sheaths.

With *sheath of the multiconductor cable grounded*, but no other grounded sheaths or ground wires in the duct bank, the zero-sequence self-impedance $Z_{00-v}$ of circuit $abc$, the current $I_v$, and the mutual impedance $Z_{0m-v}$ between circuit $abc$ and any conductor $A$ of the multiconductor cable are, respectively,

$$Z_{00-v} = Z_{aa} + 2Z_{\overline{ab}-g} - 3\frac{Z^2_{\overline{av}-g}}{Z_{vv-g}} \qquad [63]$$

$$I_v = -3I_{a0}\frac{Z_{\overline{av}-g}}{Z_{vv-g}} \qquad [64]$$

$$Z_{0m-v} = -r_v\frac{I_v}{I_{a0}} = 3r_v\frac{Z_{\overline{av}-g}}{Z_{vv-g}} \qquad [65]$$

where bars over the subscripts indicate that geometric mean distances are used in determining mutual impedances. Thus $Z_{\overline{ab}-g}$ corresponds to $s = \sqrt[3]{s_{ab}s_{ac}s_{bc}}$; $Z_{\overline{av}-g}$ corresponds to $\sqrt[3]{s_{av}s_{bv}s_{cv}}$, etc.

With a *second grounded sheath or ground wire u* in addition to grounded sheath $v$ in the duct bank, as indicated in Fig. 16,

$$I_v = -3I_{a0}\frac{Z_{\overline{av}-g}Z_{uu-g} - Z_{\overline{au}-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{uv-g}} \qquad [66]$$

$$I_u = -3I_{a0}\frac{Z_{\overline{au}-g}Z_{vv-g} - Z_{av-g}Z_{vu-g}}{Z_{vv-g}Z_{uu-g} - Z^2_{uv-g}} \qquad [67]$$

$$Z_{00-vu} = Z_{aa-g} + 2Z_{\overline{ab}-g} + \frac{I_v}{I_{a0}}Z_{\overline{av}-g} + \frac{I_u}{I_{a0}}Z_{\overline{au}-g} \qquad [68]$$

$$Z_{0m-vu} = -r_v\frac{I_v}{I_{a0}} \qquad [69]$$

With *more than one grounded sheath or ground wire* in addition to grounded sheath $v$ in the duct bank, [66]–[69] can be approximately applied if all identical sheaths including $v$ are grouped into one equivalent sheath $v$, and all identical ground wires into one equivalent ground wire $u$. $Z_{vv-g}$ and $Z_{uu-g}$ are evaluated from [57], and geometric mean distances are used in determining mutual impedances with earth return between equivalent sheath $v$ and equivalent ground wire $u$, and between conductors and equivalent sheath or ground wire.

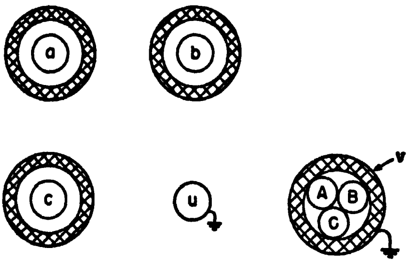

FIG. 16. Three-phase circuit *abc* of single-conductor cables with open-circuited sheaths, a multiconductor cable *ABC* with grounded sheath $v$, and a ground wire or additional grounded sheath $u$.

With *two dissimilar sheaths and a ground wire:* if it is necessary to consider one ground wire (or group of ground wires) $p$ and one grounded sheath (or group of sheaths) $u$ in addition to grounded sheath $v$ of the multiconductor cable, the currents $I_p$, $I_u$, and $I_v$ in terms of $I_{a0}$ can be calculated from the equations for currents in three dissimilar ground wires, given on page 399, Volume I, if $p$ replaces $w$ in these equations. The zero-sequence self-impedance $Z_{00-puv}$ of a three-phase circuit of single-conductor cables with no currents in their sheaths and the mutual impedances $Z_{0m-puv}$ between circuit *abc* and a conductor $A$ of the control cable with sheath $v$ are, respectively,

$$Z_{00-puv} = Z_{aa-g} + 2Z_{ab-g} + Z_{\overline{av}-g}\frac{I_v}{I_{a0}} + Z_{\overline{au}-g}\frac{I_u}{I_{a0}} + Z_{\overline{ap}-g}\frac{I_p}{I_{a0}}$$

$$Z_{0m-puv} = -r_v\frac{I_v}{I_{a0}} \qquad [70]$$

**Equivalent Circuits for A-C Network Analyzer.** When there are more sheaths and ground wires to be considered separately than can be conveniently handled by equations, solutions can be made on an a-c network analyzer if mutual coupling transformers are used. The equivalent circuit of Fig. 14(*b*) is applicable when there is a three-phase circuit *abc* of single-conductor cables with no currents in their sheaths, a grounded sheath (or group of grounded sheaths) *u*, a ground wire (or group of ground wires) *p*, and a multiconductor power or control cable *ABC* with grounded sheath *v*, if *the connection between w and ground is opened* and the internal zero-sequence self-impedance $Z'_{00(a)}$ and the impedance $Z'_a$ are replaced by a single impedance $Z''_a$ at either location, where

$$Z''_a = Z_{aa-g} + 2Z_{\overline{ab}-g} - 3(Z_{\overline{av}-g} + Z_{\overline{av}-g} - Z_{\overline{ap}-g}) \qquad [71]$$

The internal zero-sequence self-impedance $Z'_{00}(A)$ and $Z'_{00}(A')$ indicated in Fig. 14(*b*) are not required in determining the zero-sequence self-impedance of circuit *abc* nor the mutual impedance between circuit *abc* and any conductor *A* of the multiconductor cable with grounded sheath *v*.

**Example 5.** Given: A 60-cycle, three-phase circuit of 69-kv single-conductor, hollow core, oil-filled, lead-covered cables. The conductors are 1,100,000 cm stranded conductors of 1.365 inches outside diameter and 0.650 inch inside diameter. Insulation sleeves are introduced in the sheaths at manholes, the sheath being grounded on only one side of the sleeves. The cables are in separate ducts of a duct bank of eight ducts, arranged two across and four deep with 6 inches between duct centers. The three-phase circuit occupies the two first-row ducts and one second-row duct. In the other second-row duct, a multiconductor control cable with ground sheath and a 4/0 hard-drawn copper ground wire have been installed. The diameter of the control cable under the lead is 0.816 inch; lead thickness is 5/64 inch; outside diameter is 0.972 inch. The four lower ducts at present are empty, but a second identical three-phase circuit of 69-kv single-conductor cables will be installed later. It is estimated that earth resistivity $\rho$ is 100 ohms (meter cube); copper temperature of power cable is 75°C; control cable sheath temperature is 40°C; and ground wire temperature is 25°C.

(*a*) Determine the zero-sequence self-impedance of the three-phase cable circuit, the currents in sheath, ground wire and ground, and the voltage induced in a conductor of the control cable in terms of the zero-sequence current per phase in the three-phase circuit. Determine also the voltage induced in a conductor of the control cable, (*b*) with the ground wire removed, and (*c*) with the ground wire removed and the sheath of the control cable on open circuit. (*d*) Repeat (*a*), assuming that the second three-phase cable circuit has been installed.

*Solution.* (*a*) Let $a$, $b$, $c$ be the conductors of the installed three-phase cable circuit, $A$ any conductor of the control cable with sheath $v$, and $u$ the ground wire. The arrangement is similar to that of Fig. 16, except that $u$ and $v$ are in the same duct.

Equations [66]–[69] are applicable for part (*a*).

$$s_{\overline{ab}} = s_{\overline{av}} = s_{\overline{au}} = \sqrt[3]{6 \times 6 \times 6\sqrt{2}} = 6.73 \text{ inches}$$

Under the assumption that $u$ and $v$ are in contact, $s_{vu} = \frac{1}{2}(d_v + d_u) = 0.747$ inch. If the ground wire is buried in the concrete or installed in another duct, $s_{vu}$ will have a higher value.

From Volume I, page 500, $d_u$, $r_u$, and $x_i$ for the 4/0 ground wire at 25°C are $d_u = 0.522$ inch; $r_u = 0.276$ ohm per mile; $x_i = 0.039$ ohm per mile.

The d-c resistance of conductors $a$, $b$, and $c$ from [17], Chapter II, at 75°C, increased by 3% to allow for spiraling, is $R_{dc} = 0.0118$ ohm per 1000 feet. $(d_i)a = 0.650$ inch; $(d_o)_a = 1.365$ inch. From Fig. 3, Chapter II, the skin-effect resistance ratio, with $\sqrt{f/R_{dc}} = 71.5$ and $t/d = 0.26$, is 1.028. $r_a = 0.0118 \times 1.028 = 0.0121$ ohm per 1000 feet. Conductor internal reactance $x_i$ corresponding to $d_i/d_o = 0.650/1.265 = 0.48$, read from Fig. 1, Chapter II, is 0.0038 ohm per 1000 feet.

From Fig. 3, with $D_i = 0.816$ inch and a lead thickness of $\frac{5}{64}$ inch, $r_v = 0.535$ ohm per 1000 feet. From [30], $R_g = R_{aa-g} = R_{ab-g} = R_{vu-g} = 0.018$ ohm per 1000 feet.

Self- and mutual reactances with earth return may be determined from Fig. 8, or calculated from [31] and [32]. The latter method was used in determining the values given below, where all impedances are in ohms per 1000 feet.

$$Z_{aa-g} = (0.0121 + 0.0180) + j(0.2481 + 0.0038) = 0.0301 + j0.2519$$

$$Z_{vv-g} = (0.535 + 0.0180) + j0.2559 = 0.553 + j0.2559 = 0.609\underline{/24.83^\circ}$$

$$Z_{uu-g} = \left(\frac{0.276}{5.28} + 0.0180\right) + j0.2702 = 0.0703 + j0.2702$$

$$Z_{\overline{ab}-g} = Z_{\overline{av}-g} = Z_{\overline{au}-g} = 0.0180 + j0.1955 = 0.1963\underline{/84.74^\circ}$$

$$Z_{vu-g} = 0.0180 + j0.2461$$

With ground wire $u$ and control cable with sheath $v$ in the same duct, $Z_{\overline{av}-g} = Z_{\overline{au}-g}$. In this particular example, $Z_{\overline{ab}-g} = Z_{\overline{av}-g} = Z_{\overline{au}-g}$.

$$Z_{uu-g} - Z_{vu-g} = 0.0523 + j0.0241 = 0.0576\underline{/24.74^\circ}$$

$$Z_{vv} - Z_{vu-g} = 0.535 + j0.0098 = 0.535\underline{/1.05^\circ}$$

$$Z_{uu-g}Z_{vv-g} = -0.0303 + j0.1674$$

$$Z^2_{wv-g} = -0.0603 + j0.0089$$

$$Z_{uu-g}Z_{zz-g} - Z^2_{wv-g} = 0.0300 + j0.1585 = 0.1613\underline{/79.28^\circ}$$

From [66] and [67], with $Z_{\overline{au}-g} = Z_{\overline{av}-g}$, $I_v$ and $I_u$ are

$$I_v = -3I_{a0}\frac{(0.1963\underline{/84.74^\circ})(0.0576\underline{/24.74^\circ})}{0.1613\underline{/79.28^\circ}} = -3I_{a0}(0.0701\underline{/30.20^\circ})$$

$$= -3I_{a0}(0.0606 + j0.035)$$

$$I_u = -3I_{a0}\frac{(0.1963\,\underline{/84.74^\circ})(0.535\,\underline{/1.05^\circ})}{0.1613/79.28^\circ} = -3I_{a0}(0.651\,\underline{/6.51^\circ})$$

$$= -3I_{a0}(0.647 + j0.074)$$

$$I_g = -(3I_{a0} + I_v + I_u) = -3I_{a0}(0.292 - j0.109) = -3I_{a0}(0.312\,\underline{/-20.5^\circ})$$

From [68] and [69], the zero-sequence self-impedance $Z_{00-vu}$ and the mutual impedance $Z_{0m-vu}$ between circuit *abc* and conductor *A* are

$$Z_{00-vu} = Z_{aa-g} + 2Z_{\overline{ab}-g} + \frac{(I_w + I_v)}{I_{a0}} Z_{\overline{av}-g}$$

$$= 0.0661 + j0.643 - 3(0.708 + j0.109)\ (0.0180 + j0.1955)$$

$$= 0.0661 + j0.643 - (-0.0257 + j0.421) = 0.092 + j0.222$$

$$Z_{0m-wv} = -r_v\frac{I_v}{I_{a0}} = 3 \times 0.535 \times 0.0701\,\underline{/30.2^\circ} = 0.1125\,\underline{/30.2^\circ}$$

The voltage to ground $V_{A0}$ induced in any conductor *A* of the control cable with ground sheath *v* and a ground wire *u* in the same duct bank is

$$V_{A0} = I_{a0}Z_{m0-vu} = I_{a0}\ (0.1125\,\underline{/30.2^\circ}) \text{ volts per 1000 feet of conductor } A$$

where $I_{a0}$ is the zero-sequence current per phase in the three-conductor cable.

(*b*) With the ground wire removed, $Z_{0m-v}$ given by [65] is

$$Z_{0m-v} = 3r_v\frac{Z_{\overline{av}-g}}{Z_{\overline{vv}-g}} = \frac{3\ (0.535)\ 0.1963\,\underline{/84.74^\circ}}{0.609\,\underline{/24.83^\circ}} = 0.517\,\underline{/59.9^\circ}$$

The voltage-to-ground $V_{10}$ induced in conductor *A* of the control cable is

$$V_{A0} = I_{a0}Z_{0m-v} = I_{a0}(0.517\,\underline{/59.9^\circ}) \text{ volts per 1000 feet}$$

(*c*) With no ground wire and the sheath of the control cable on open circuit, the mutual impedance $Z_{0m}$ between circuit *abc* and conductor *A* is

$$Z_{0m} = 3Z_{\overline{aA}-g} = 3Z_{\overline{av}-g} = 3\ (0.1963/84.74^\circ) = 0.589/84.7^\circ$$

The voltage to ground $V_{a0}$ induced in conductor *A* of the control cable is

$$V_{A0} = I_{a0}Z_{0m} = I_{a0}(0.589\,\underline{/84.7^\circ}) \text{ volts per 1000 feet}$$

(*d*) As the second three-phase circuit is operated with sheaths on open circuit, it will have no effect upon the calculations made under (*a*), unless an additional control cable with or without a second ground wire is installed. If this is the case, as an approximation, the two control cables can be replaced by one equivalent control cable and the two ground wires (if there are two) by one equivalent ground wire and [66]–[69] can be applied, as explained below these equations. The *average* self-impedance of the two power circuits and the *average* mutual impedance between power circuits and conductors of the control cables will be obtained if both power circuits are considered in determining geometric mean distances between power circuits and equivalent control cable, and between power circuits and equivalent ground wire.

**2. Currents in the Sheaths of Single-Conductor Cables.** (See "Methods of Sheath Operation" above.) When the single-conductor cable sheaths are operated short-circuited (bonded and grounded), or when operated open-circuited with cross bonding and the sheaths bonded and grounded at the end of each completed transposition, zero-sequence currents can flow in the sheaths. If impedance bonds or bonding transformers are used, zero-sequence sheath currents are reduced in magnitude from their values obtained with sheaths short-circuited, or open-circuited with cross bonding. This effect can be included in zero-sequence impedance calculations by adding the impedances of the impedance bonds or the leakage impedances of the bonding transformers to the self-impedances with earth return of the sheaths.

**Three-Phase Circuits with Sheaths Short-Circuited (Bonded and Grounded).** In a three-phase circuit of identical single-conductor cables, let $a$, $b$, $c$ represent the phase conductors and $x$, $y$, $z$ their respective sheaths which have equal resistances $r_x$. In a symmetrical circuit, with only zero-sequence currents in the phase conductors, the sheath currents will be equal. This is approximately true in an unsymmetrical circuit. If it is assumed that the sheaths carry equal currents, they may be replaced by one equivalent sheath $w$, with self-impedance $Z_{ww-g}$ and equivalent mutual impedance $Z_{aw-g}$ with the phase conductors, where

$Z_{ww-g} = \frac{1}{3}(Z_{xx-g} + 2Z_{\overline{xy}-g})$ = self-impedance with earth return of equivalent sheath $w$ [72]

$Z_{aw-g} = \frac{1}{3}(Z_{ax-g} + 2Z_{\overline{ay}-g})$ = mutual impedance with earth return between phase conductors and equivalent sheath $w$ [73]

$Z_{xx-g}$ = self-impedance with earth return of sheaths $x$, $y$, or $z$

$Z_{ax-g}$ = mutual impedance with common earth return between any conductor and its concentric sheath

$Z_{\overline{xy}-g} = Z_{\overline{ay}-g}$ = average mutual impedance with common earth return between sheaths $x$, $y$, $z$, or between conductors $abc$ and the sheaths $x$, $y$, and $z$ which are non-concentric, and corresponds to the *gmd* between sheaths

Since $Z_{xx-g} = r_x + Z_{ax-g}$ and $Z_{\overline{ay}-g} = Z_{xy-g}$, $Z_{ww-g}$, and $Z_{aw-g}$ in [72] and [73] may be written

$$Z_{ww-g} = \left(\frac{r_x}{3} + R_g\right) + j\tfrac{1}{3}(X_{xx-g} + 2X_{\overline{xy}-g}) = r_w + R_g + jX_{ww-g} \quad [74]$$

$$Z_{aw-g} = R_g + \tfrac{1}{3}(X_{xx-g} + 2X_{xy-g}) = R_g + X_{ww-g} = Z_{ww-g} - r_w \quad [75]$$

where $r_w = r_x/3$ and $X_{ww-g} = \frac{1}{3}(X_{xx-g} + 2X_{\overline{xy}-g})$, and $X_{xx-g}$ and $X_{\overline{xy}-g}$ are given by [31] and [32], respectively, if subscripts $x$ and $y$ replace $a$ and $b$ in these equations.

With three single-conductor cables replaced by one equivalent cable with sheath $w$, where the self-impedance $Z_{ww-g}$ of the sheath with earth return and the mutual impedance $Z_{aw-g}$ between sheath and conductors with common earth return are given by [74] and [75], respectively, the equations developed by three conductor cables with grounded sheaths can be applied. In applying these equations, the *gmd* between the axis of any sheath or conductor and the axes of sheaths $x$, $y$, $z$ are used to determine mutual impedances with earth return between that sheath or conductor and circuit *abc* or equivalent sheath $w$.

**Single-Phase Circuits.** The sheaths $x$ and $y$ of two single-conductor cables $a$ and $b$ with sheaths short-circuited can be treated as one equivalent sheath $w$ of self-impedance $Z_{ww-g} = \frac{1}{2}(Z_{xx-g} + Z_{xy-g})$ and mutual impedance with the phase conductors $Z_{aw-g} = \frac{1}{2}(Z_{ax-g} + Z_{ay-g})$. With the two grounded sheaths replaced by one equivalent sheath, zero-sequence impedances of single-phase cable circuits are calculated just as single-phase overhead transmission circuits with ground wires.

## CAPACITANCE OF UNDERGROUND CABLES WITH LEAD SHEATHS

The capacitance $C$ between an insulated round wire within a concentric sheath and the sheath is[17]

$$C = \frac{7.354 \times 10^{-3}K}{\log_{10}\dfrac{D_i}{d}} \text{ microfarads per 1000 feet} \quad [76]$$

where $K$ is the specific inductive capacity (S.I.C.) of the insulation; $d$ is the diameter of the conductor; $D_i$ is the inside diameter of the sheath or the outside diameter of the insulation if shielding tape is used. If the sheath is grounded, [76] gives the capacitance to ground.

**Single-Conductor Cables.** The positive-, negative-, and zero-sequence capacitances $C_{11}$, $C_{22}$, and $C_{00}$, respectively, of a three-phase circuit and the positive- and zero-sequence capacitances $C_{11}$ and $C_{00}$ of a single-phase or two-phase circuit of single-conductor cables with grounded sheaths are the same as the capacitance of one of the conductors to its sheath given by [76].

**Type H Cables.** As the shielding tape around the conductors is in contact with the binder tape which is in direct contact with the sheath, the shielding tape and sheath are at the same potential. For determining capacitances, the three-conductor type H cable may be treated as three single-conductor cables having the same conductor diameters and insulation thicknesses; $C_{11} = C_{22} = C_{00}$ can therefore be determined from [76], if $d$ is conductor diameter and $D_i = d + 2T$, where $T$ is thickness of conductor insulation. Likewise, for the two-conductor type H cable, $C_{11} = C_{00}$ is determined from [76].

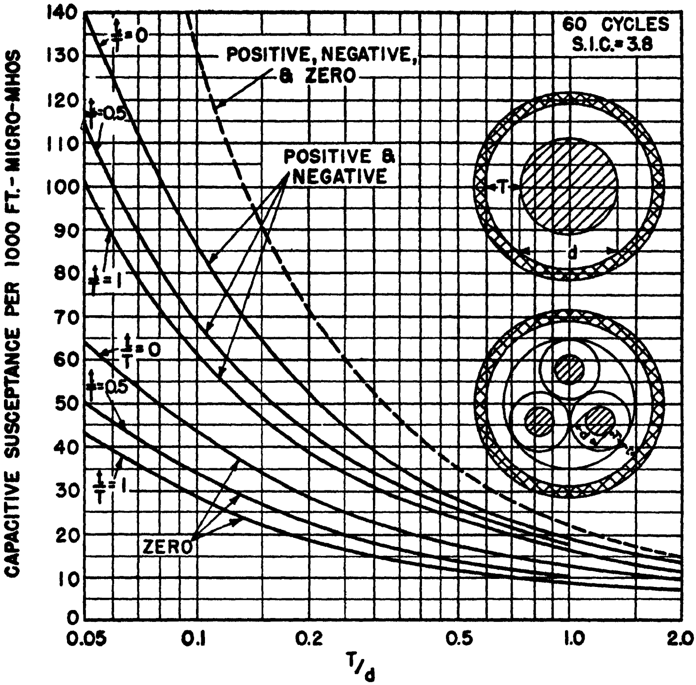

FIG. 17. Capacitive susceptance of cables.

$$b = 2\pi fC = \frac{f}{60} \times \frac{\text{S.I.C.}}{3.8} \times \text{(value read from chart)}$$

The broken curve in Fig. 17 gives the positive-, negative-, and zero-sequence capacitive susceptance in micromhos per 1000 feet of three-phase circuits of single-conductor or type H three-conductor cables with round conductors, calculated from [76] at a frequency of 60 cps and a specific inductive capacity (S.I.C.) of $K = 3.8$. At any other frequency or specific inductive capacity, the value read from the curve is to be multiplied by $(f/60) \times (K/3.8)$. (See Table I for approximate values of $K$.) This curve can be used also to determine the positive- and zero-sequence capacitive susceptance of single-phase and two-phase circuits of single-conductor or two-conductor type H cables.

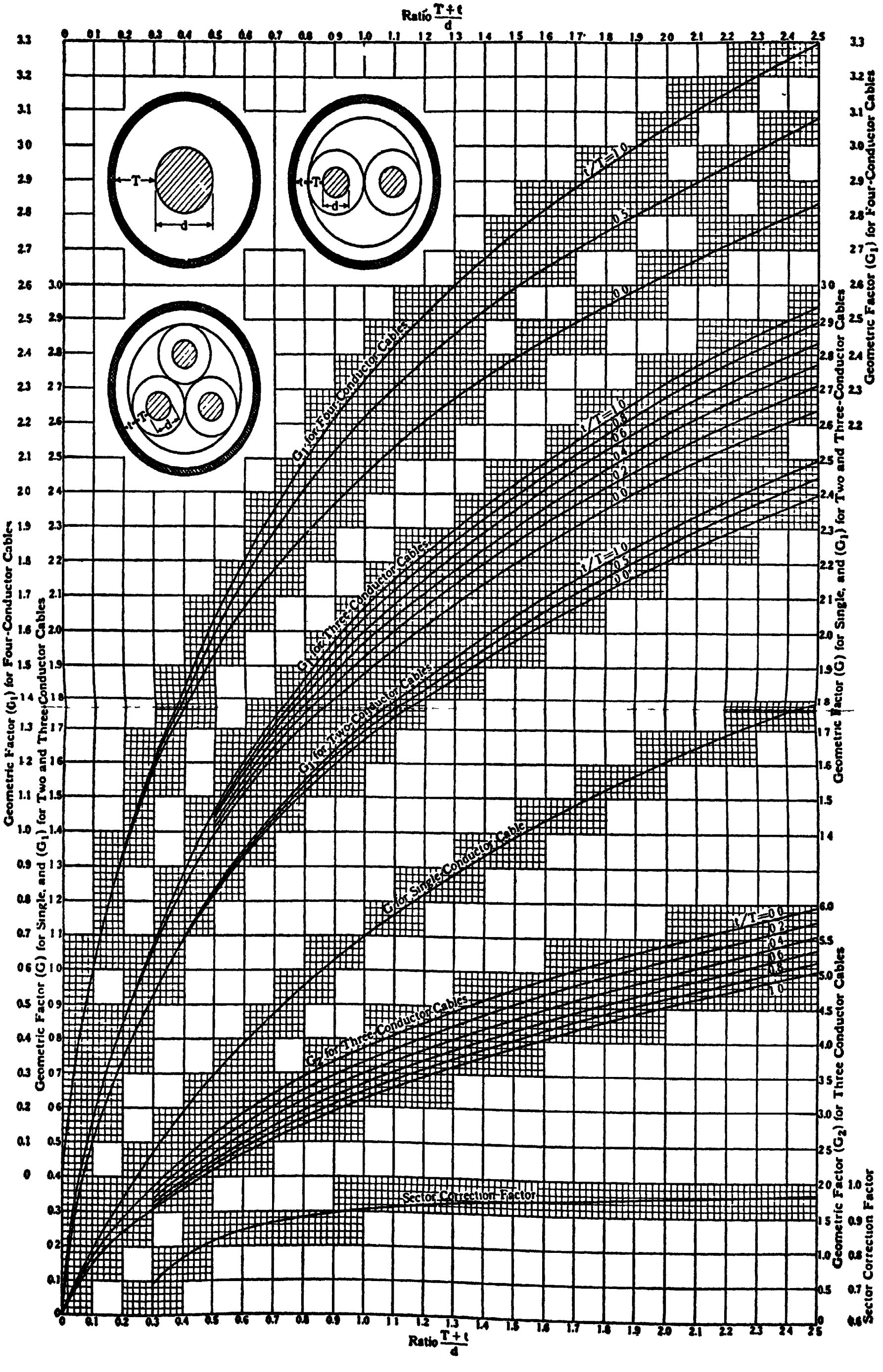

**Fig. 18.** Geometric factors. $G_1$ indicates zero-sequence and $G_2$ positive-sequence geometric factors. (References 2 and 8.)

For *three-conductor type H cables with sector conductors*, the capacitances are approximately those of a cable with round conductors of the same circular mils and the thickness of conductor insulation.

**Three-Conductor Belted Cables.** Two insulation thicknesses are to be considered in belted cable, the conductor insulation of thickness $T$ and the belt of thickness $t$. Figure 18, taken from the *Underground System Reference Book*,[8] page 297, gives *geometric factors* versus $(T + t)/d$ with $t/T$ as parameter for belted cables with round conductors (and corrections for sector conductors) from which their positive-, negative-, and zero-sequence capacitances can be determined. The general capacitance equation[8] is

$$C = \frac{0.0169nK}{G} \text{ microfarads per 1000 feet of cable} \qquad [77]$$

where $K$ = specific inductive capacity (S.I.C.)
$n$ = number of conductors
$G$ = geometric factor from Fig. 18.

The geometric factor for single-conductor and multiconductor shielded cables is indicated in Fig. 18 by $G$. Instead of using [76] for determining the positive-, negative-, and zero-sequence capacitances of single-conductor and type H cables, [77], with $n = 1$ and $G$ read from Fig. 18, could have been used as well.

Zero-sequence geometric factors for two-, three-, and four-conductor belted cables are indicated by $G_1$; positive-sequence geometric factors for three-conductor cables by $G_2$. Positive-, negative-, and zero-sequence capacitive susceptances of three-conductor belted cables with *round conductor*, calculated by means of these geometric factors, are given by the solid curves of Fig. 17 for a frequency of 60 cps and $K = 3.8$. The correcting factor to be applied to the values read from the solid curves of Fig. 17, to obtain capacitive susceptances of belted cables with sector conductors of the same circular mils and insulation thicknesses is the reciprocal of the sector correction factor given in Fig. 18. (Note that the abscissa in Fig. 18 is not the same as that in Fig. 17.)

**Example 6.** Determine the positive- and zero-sequence capacitive susceptances $b_1$ and $b_0$ in micromhos per 1000 feet of the following cables in a three-phase system with grounded neutral.

(*a*) 34.5-kv, 50-cycle, three-conductor, type H (shielded) cable with 500,000-cir mils copper sector conductors and solid paper insulation.

(*b*) 15-kv, 60-cycle, three-conductor belted cable with 350,000-cir mils copper round conductors and varnished-cambric insulation.

(*c*) Repeat (*b*) with sector conductors.

*Solution.* The approximate values of S.I.C. from Table I and the insulation thickness of Appendix A will be used.

(*a*) From Tables III and IX, Appendix A: $d$ (of equivalent round conductor) = 0.815 inch, $T = 0.375$ inch. $T/d = 0.46$ inch. At 60 cps and S.I.C. = 3.8, from the broken curve of Fig. 17, $b_{11} = b_{00} = 37$ micromhos per 1000 feet. At 50 cps and S.I.C. = 4.0,

$$b_{11} = b_{00} = \frac{50}{60} \times \frac{4.0}{3.8} \times 37 = 32.5 \text{ micromhos per 1000 feet}$$

The capacitive susceptance of sector conductor in three-conductor type H cables is approximately that of round conductors of the same circular mils and insulation thickness.

(*b*) From Tables III and VII, Appendix A: $d = 0.682$ inch; $T = 0.203$ inch; $t = 0.109$ inch; $T/d = 0.298$; $t/T = 0.538$. From Table I, the S.I.C. for varnished cambric is 5. Values read from the solid curves of Fig. 17, multiplied by 5/3.8, are

$$b_1 = 34 \times \frac{5}{3.8} = 44.7 \text{ micromhos per 1000 feet}$$

$$b_0 = 17.5 \times \frac{5}{3.8} = 23.1 \text{ micromhos per 1000 feet}$$

(*c*) From (*b*), $(T + t)/d = 0.458$.

From Fig. 18, sector correction factor = 0.80.

$$b_1 = 44.7/0.80 = 56 \text{ micromhos per 1000 feet}$$

$$b_0 = 23.1/0.80 = 29 \text{ micromhos per 1000 feet}$$

**Example 7.** Determine the dielectric loss under normal balanced operation of the cable described in part (*b*) of Example 6 at 70°C.

*Solution.* From Table I, the 60-cycle power factor of varnished-cambric insulation in belted cable at 70°C is 12.5%.

For substitution in [6] or [6*a*],

$$V_{a1} = \text{positive-sequence voltage to neutral} = \frac{15{,}000}{\sqrt{3}} = 8660 \text{ volts}$$

From Example 6, $b_1 = (2\pi f C_1) = 44.7 \times 10^{-6}$ mhos.

$$\cos\theta = 0.125, \quad \cot\theta = 0.126$$

Substitution of the above values in [6] and [6*a*], respectively, gives

$$W = 3|8660|^2\,(44.7 \times 10^{-6}) \times 0.126 = 127 \text{ watts per 1000 feet}$$

$$W = 3|8660|^2\,(44.7 \times 10^{-6}) \times 0.125 = 126 \text{ watts per 1000 feet}$$

The following values are required for substitution in [6*b*]:

$e = 8.66$ kv; $f = 60$; $n = 3$; $K = 5$; $\cos\theta = 0.125$

$G_2$ [from Fig. 18 corresponding to $(T + t)/d = 0.458$ and $t/T = 0.538$] = **2.13**

From [6*b*] for 1000 feet of three-conductor cable,

$$W = \frac{0.106 \times (8.66)^2 \times 60 \times 3^2 \times 5 \times 0.125}{2.13} = 125 \text{ watts per 1000 feet}$$

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# CHAPTER IV

## TRANSFORMERS AND AUTOTRANSFORMERS

The apparatus which has largely contributed to the flexibility and adaptability of alternating-current supply, and is responsible for its successful application, is the transformer. It enables the voltage to be changed very easily to that best suited to the requirements encountered in each part of the power system. Alternating-current electric power can be generated at moderate voltage (usually 13.8 kv for large machines) so as not to require a large ratio of insulation to copper, then stepped up to ten or twenty times this voltage for economical transmission to distant load centers. Normally, a power system consists of several networks at different voltage levels, the networks being connected to each other by transformers or autotransformers. From the high-voltage main transmission lines, the voltage may be reduced in successive steps to a lower-voltage transmission network, a " primary " distribution network, and finally to the users' circuits.

The transformer also provides a convenient means for electrically insulating from one another various parts of a system. Two utility companies may be interconnected for purposes of selling or buying power or for emergency support, without requiring similar methods of neutral grounding. The methods of grounding at the different voltage levels of any one system need not necessarily be the same, although proper system protection requires that the various grounding methods and protective equipment be coordinated.

A transformer accomplishes its dual function by close electromagnetic coupling between circuits. As the transformer has ordinarily no moving parts, there need be no compromise between best magnetic characteristics and highest mechanical strength in the choice of steel for the core; steel which keeps the required magnetizing current and iron losses associated with the coupling to a minimum can be selected, and the coils can be insulated with shields and fluid insulation which provide a higher breakdown strength than can readily be attained with other types of apparatus.

### Fundamental Theory

For two electromagnetically coupled coils, if iron saturation and losses are neglected, the two voltage equations may be written

$$\begin{aligned} e_1 &= L_{11}\frac{d}{dt}i_1 + r_1 i_1 + L_{12}\frac{d}{dt}i_2 \\ e_2 &= L_{12}\frac{d}{dt}i_1 + L_{22}\frac{d}{dt}i_2 + r_2 i_2 \end{aligned} \qquad [1]$$

where subscripts 1 and 2 refer to coils 1 and 2, respectively, $e_1$ and $e_2$ are instantaneous voltages across the coils, $i_1$ and $i_2$ instantaneous currents in the coils, $r_1$ and $r_2$ coil resistances, $L_{11}$ and $L_{22}$ coil self-inductances, and $L_{12}$ mutual inductance between the two coils. For sinusoidally alternating current, where the rate of change of current $(di/dt)$ is $2\pi f$ times the current, shifted in phase by 90°, (see [52]–[54], Chapter II), the two circuit equations may be written vectorially as

$$\begin{aligned} V_1 &= (r_1 + jx_{11})I_1 + jx_{12}I_2 \\ V_2 &= jx_{12}I_1 + (r_2 + jx_{22})I_2 \end{aligned} \qquad [2]$$

where $V$ and $I$ indicate vector voltage and current, respectively, and $x = 2\pi fL$.

The total flux linking the transformer coils consists of mutual flux and leakage flux. Mutual flux links both coils. The leakage flux of either coil is that portion of the flux linking the coil which does not also link the other coil. The rate of change of mutual flux determines the induced voltages in the coils which are in direct proportion to the number of turns in the coils. Voltages induced by mutual flux are called *induced voltages* as distinguished from terminal or applied voltages.

With coils 1 and 2 having turns $N_1$ and $N_2$, respectively, on the same iron core, the path of the mutual flux which links both coils is in the iron. The permeance $P_{12} = P_{21}$ of the mutual flux path is therefore the permeance of the iron core. Under normal operation, the leakage flux associated with either coil is very small relative to the mutual flux. The paths of the leakage fluxes, being largely in air, have approximately constant permeances. The corresponding leakage reactances at constant frequency are therefore substantially constant. To simplify the problem, the resistance $r_1$ and the leakage reactance $x_1$ of coil 1 may be treated as an external impedance $r_1 + jx_1$ in series with the coil; similarly, the resistance $r_2$ and the leakage reactance $x_2$ of coil 2 may be considered an external impedance $r_2 + jx_2$ in series with coil 2. With this arrangement, the permeances $P_{11}$ and $P_{22}$ of coils 1 and 2, respectively, with the permeance of the leakage flux paths neglected, are the permeances of the iron core;

they are therefore equal to the mutual permeance $P_{12}$. From [48] and [49] of Chapter II, with $P_{11} = P_{22} = P_{12}$, the self-inductance minus the leakage inductance in coils 1 and 2 and the mutual inductance between the coils, when expressed in henries, are in the proportion $N_1^2 : N_2^2 : N_1N_2$.

Let all quantities be expressed in per unit, based on a common kva with base or unit voltages in the two coils in direct proportion to their turns. Based on the same kva, the per unit self-impedance in any circuit varies inversely as the square of the base voltage in that circuit, and the per unit mutual impedance between two circuits inversely as the product of the base voltages in the two circuits. Therefore, in per unit

$$x_{11} - x_1 = x_{22} - x_2 = x_{12}$$

Equations [2], with $x_{12}$ replaced by $x_m$, subscripts 1 and 2 replaced by $p$ and $s$ to indicate primary and secondary windings, respectively, positive direction of $I_s$ opposite to $I_p$ in the coils, and all quantities in per unit, may be written

$$\begin{aligned} V_p &= (r_p + jx_p)I_p + jx_m(I_p - I_s) \\ V_s &= jx_m(I_p - I_s) - (r_s + jx_s)I_s \end{aligned} \qquad [3]$$

where $r_p$ and $r_s$ are resistances, and $x_p$ and $x_s$ are leakage reactances in primary and secondary windings, respectively, and $x_m$ is mutual reactance between windings.

The per unit equivalent circuit which satisfies equations [3] is shown in Fig. 1(*a*), with the addition of the resistance component $R_{h+e}$ of the mutual branch which depends upon hysteresis and eddy-current losses in the iron core. In Fig. 1(*a*), $V_p$ and $V_s$ are per unit voltages, and $I_p$ and $I_s$ are per unit currents in primary and secondary windings, respectively; $V_i$ is the per unit voltage induced in both windings by the mutual flux.

**Components of Mutual Impedance.** At constant applied frequency, iron losses vary approximately as the square of the induced voltage. They can therefore be taken into account in an equivalent circuit by placing a resistance branch in parallel with the magnetizing branch. In Fig. 1(*a*), the resistance branch $R_{h+e}$ of the mutual impedance parallels the magnetizing impedance $jx_m$. The power component of the exciting current flows through the resistance branch $R_{h+e}$ and is in phase with the induced voltage; the magnetizing component of the exciting current flows through the magnetizing impedance $jx_m$ and is in quadrature with the induced voltage; both components of exciting current flow through the resistance and leakage reactance of the winding supplying the exciting current.

At no load and rated voltage, the exciting current flowing in the closed winding causes small resistance and leakage-reactance voltage drops in that winding; these voltage drops at no load, however, are negligible relative to the applied rated voltage. Therefore the per unit mutual impedance at rated voltage may be considered equal to the per unit exciting or self-impedance of the winding at rated voltage.

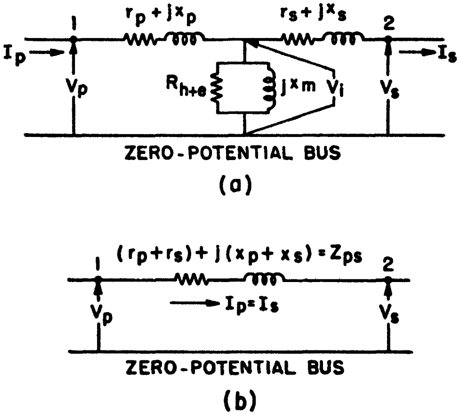


FIG. 1. Per unit equivalent circuits for two-winding transformer. (*a*) Exciting current included. (*b*) Exciting current neglected.

In determining mutual impedance from exciting impedance at rated voltage by test, it is immaterial which winding is used; expressed in per unit on the same kva base and with base voltages directly proportional to the number of turns in the windings, the exciting impedance (neglecting the small no-load resistance and leakage-reactance voltage drops) is the same referred to either winding. Because of voltage limitations in testing equipment, the low-voltage winding is the one usually tested for the exciting impedance. By measuring watts and current with rated voltage applied at rated frequency, both the magnetizing component and the power component of the exciting impedance can be obtained at rated voltage and frequency.

**Mutual Impedance Infinite.** There are many system problems in which the required degree of precision is such that exciting currents can be neglected, in short-circuit calculations, for example. Stated in another way, the mutual impedances between transformer windings are so large relative to the other impedances in the system that they can be considered relatively infinite for determining currents and voltages within the required degree of precision. In such cases, the equivalent circuit of Fig. 1(*a*) reduces to that of Fig. 1(*b*) in which there is a single series impedance $Z_{ps} = (r_p + r_s) + j(x_p + x_s)$ between primary and secondary terminals. This impedance, which is the sum of the resistances and leakage reactances of the two windings, is usually called the *leakage impedance.* It is also called the *short-circuit impedance* between windings; the mutual impedance at rated voltage being very high relative to the leakage impedances, their sum can be determined by applying a voltage to one winding with the other winding short-circuited. When mutual impedance is assumed infinite, leakage impedance is the same referred to either winding if expressed in per unit, the kva base being common to both windings and the respective voltage bases being in direct proportion to the number of turns.

**Exciting Currents and Impedances.** Although there are many conditions under which exciting currents can be neglected, and self-impedances of windings and mutual impedances between windings considered infinite relative to the other system impedances involved, there are also conditions under which this is not the case. It is then necessary to consider exciting currents and impedances and the effects of various degrees of saturation upon them. As the exciting or self-impedance of a winding is determined with all other windings open, it is often called the open-circuit impedance.

*Exciting Currents.* When a sinusoidal voltage is applied to one winding of a single-phase transformer, with the other winding or windings open, the exciting current consists of fundamental-frequency and odd harmonic terms of magnitudes depending upon the characteristics of the iron and the maximum flux density at which it is operated. In general, the higher the flux density, the higher the ratios of harmonics to fundamental, the ratio of third to fundamental being the highest.

In transformer banks of three single-phase units or in three-phase transformers under balanced operation, third harmonics and their multiples are equal in magnitude and in phase in the three phases, and are therefore zero-sequence quantities. The other harmonics are positive- or negative-sequence quantities, as they differ in time phase by 120° in the three phases.[1] If the primary or secondary windings of a three-phase transformer or transformer bank are connected in Δ, third-harmonic exciting currents (and their multiples) will flow in the Δ because of its low third-harmonic zero-sequence impedance; this will tend to prevent the appearance of third harmonic currents and voltages (and their multiples) in the transmission line supplying the transformers. The other odd harmonic currents, however, will be present. Inductive coordination measures may be required when communication circuits parallel transmission lines because of harmonic exciting currents. But, in a power system study where currents and voltages of fundamental frequency are required, harmonics in transformer exciting currents being but a part of a normally small current (itself often neglected) need not be considered unless exciting currents themselves are of importance. With harmonics neglected, the rms value of the exciting current is approximately fundamental-frequency exciting current. Values of rms exciting current at rated voltage between 3 and 5% of rated current are common; but exciting currents as high as 10% or as low as 1% of rated current may occur.[1]

*Transformer Excitation Curves.* The magnetic reluctance of the path of the mutual flux in iron is not constant but depends upon the degree of saturation in the iron. There are two types of saturation curves for a given transformer. One, called the d-c magnetization curve, gives instantaneous currents versus flux. It is similar to the curve of peak current versus peak applied sinusoidal voltage. This type of saturation curve is used in determining instantaneous currents. The other type, used where fundamental-frequency currents and voltages only need be considered, gives rms exciting currents versus rms applied sinusoidal voltages. The latter type for power transformers (transformers, for example, larger than 500 kva) is illustrated in the saturation curves of Fig. 2, where ordinates are fundamental-frequency voltages in per unit of rated voltage, applied to one winding with the other winding or windings open. The abscissas are rms exciting currents in per unit of rms exciting current at rated voltage.

Abscissas in Fig. 2 multiplied by per cent exciting current at rated voltage give exciting currents in per cent of rated current. For example, if exciting current at rated voltage is 5% of rated current, at 140% voltage and with excitation curve *A* of Fig. 2 it will be $9.5 \times 5 = 47.5\%$ of rated current.

Curve *A* of Fig. 2 is similar to the excitation curves of many large power transformers now in service which were installed before the development of transformers of the new strip steels. The exciting current at rated voltage for such transformers may be from 2.5 to 5% of rated current. Smaller power transformers, operated at the same flux densities as that of curve *A*, will have higher exciting currents at rated voltage.

Curve *B* of Fig. 2 is a typical excitation curve of the more recently developed power transformers of the new strip steels which are operated at maximum flux densities appreciably higher than the older steels. The exciting current at rated voltage of such transformers is of the order of 1% of rated current.

*Exciting Impedance.* Because of the variation of maximum flux density in the iron core with applied voltage, the fundamental-frequency equivalent circuit of Fig. 1(*a*) is not the same at all voltages. Test data from which the equivalent circuit can be constructed are usually given at rated voltage. The resistances and leakage reactances of the windings are approximately the same at all voltages. On the basis of the assumption that core losses vary approximately as the square of the induced voltage, the resistance branch $R_{h+e}$ of the mutual impedance can be assumed constant at the value determined at rated voltage. This assumption, however, does not hold when the iron is fully saturated, because the core loss is then less than that which corresponds to the square of the induced voltage and a constant $R_{h+e}$ determined at rated voltage. However, the variation of $R_{h+e}$ with voltage has much less influence on the magnitude of the exciting impedance than the variation of the magnetizing reactance $x_m$. From the transformer excitation curve (similar to those given in Fig. 2) and the per unit exciting current at rated voltage, the ratio of the applied rms sinusoidal voltage to the rms exciting current at any applied voltage can be obtained. Neglecting harmonics, this ratio gives the per unit magnitude of the fundamental-frequency exciting

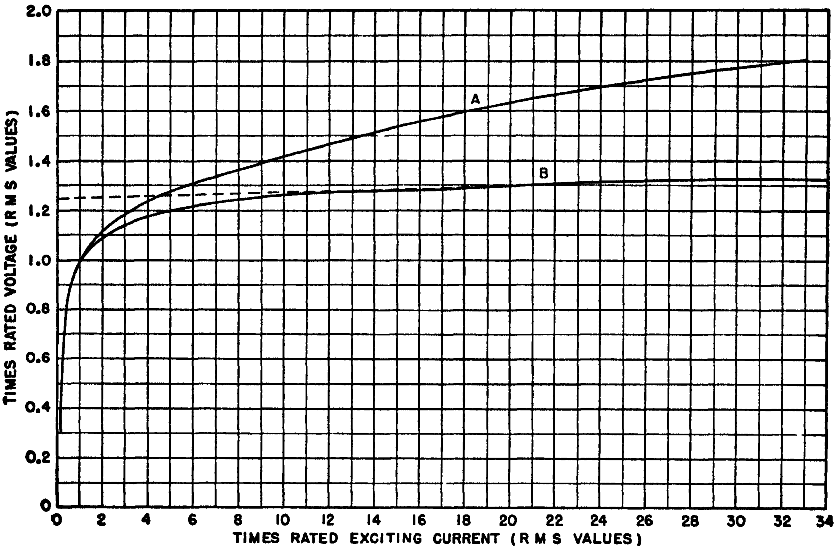


FIG. 2. Transformer saturation curves.

or self-impedance of the winding. When the power factor of the exciting impedance at rated voltage is given or can be assumed, $x_m$ at any voltage can be calculated as illustrated in Problem 1.

When the power factor of the exciting current is not given, it may be taken as approximately 20% at rated voltage and frequency for power transformers at the usual operating flux densities, and at very high flux densities as approximately 25%.

**Problem 1.** The exciting current of a two-winding transformer at rated voltage is 5% of rated current; its saturation curve is given by curve $A$ of Fig. 2. What is its approximate per unit magnetizing reactance $x_m$ (*a*) at rated voltages and (*b*) at 1.50 times rated voltage? Assume a power factor of 20% for the exciting current at rated voltage. (*c*) Repeat (*a*) and (*b*) using curve $B$ of Fig. 2, with 1% exciting current and 25% power factor at rated voltage.

*Solution.* (*a*) Let $I_{ex}$ = exciting current in per unit of rated current. With all quantities in per unit based on the transformer rating,

$$V_p = 1.0$$

and

$$I_{ex} = 0.05\underline{/-\cos^{-1} 0.20} = 0.05\underline{/-78.5^\circ} = 0.010 - j0.049$$

With $r_p$ and $x_p$ of the primary winding (which are small relative to $R_{h+e}$ and $x_m$) neglected, per unit $R_{h+e}$ and $x_m$ are

$$R_{h+e} = \frac{1.00}{0.01} = 100 \quad \text{and} \quad x_m = \frac{1.00}{0.049} = 20.4$$

(*b*) At 1.50 times rated voltage, from curve $A$ of Fig. 2,

$$I_{ex} = 0.05 \times 13.5 = 0.675 \text{ per unit of rated current}$$

If $R_{h+e}$ is assumed constant at 100 per unit,

$$I_{h+e} = \frac{1.50}{100} = 0.0150$$

$$I_m = \sqrt{(0.675)^2 - (0.015)^2} = 0.675, \text{ approximately}$$

$$x_m = \frac{1.50}{0.675} = 2.22 \text{ per unit}$$

(*c*) For curve $B$ of Fig. 2, at rated voltage with all quantities in per unit,

$$I_{ex} = 0.01\underline{/-\cos^{-1} 0.25} = 0.0025 - j0.00968$$

$$R_{h+e} = \frac{1.00}{0.0025} = 400; \quad x_m = \frac{1.00}{0.00968} = 103.3$$

Curve $B$ of Fig. 2 has not been extended to 1.50 times rated voltage; but at approximately 1.30 per unit voltage, the iron is fully saturated and the curve becomes a straight line of approximate slope $(1.30 - 1.25)/22$. At 1.50 per unit voltage,

$$I_{ex} = 0.01\left[22 + (1.50 - 1.30)\frac{22}{1.30 - 1.25}\right] = 1.10$$

As $I_{ex}$ is largely magnetizing current [see (*b*) above], $x_m$ is approximately

$$x_m = \frac{1.50}{1.10} = 1.36 \text{ per unit}$$

Problem 1 illustrates the variation in transformer magnetizing reactance with voltage. It will be noted that at 1.50 per unit voltage and the rated exciting currents and excitation curves considered, the magnetizing reactance $x_m$ remains fairly large relative to the leakage reactances whose sum is usually between 0.05 and 0.15 per unit. At very much higher values of voltage, the leakage reactance of the winding supplying the exciting current is a more important part of the exciting impedances and should be considered in determining magnetizing reactance, if the self-reactance is to be divided into leakage and magnetizing reactance.

The total per unit self-reactance $x_{pp}$ (leakage plus magnetizing reactance) of a winding with other windings open at a voltage $E$ above the voltage at which full saturation occurs is given approximately by the equation

$$x_{pp} = x_m + x_p = \frac{E(E_s - E_0)}{(E - E_0)sI_{ex}} \qquad [4]$$

where $I_{ex}$ = per unit exciting current at rated voltage

$E$ = given per unit voltage (above saturation voltage) not shown on excitation curve

$E_s$ = any per unit voltage shown on excitation curve at which this curve has become a straight line

$s$ = ratio of exciting current at $E_s$ to rated exciting current

$E_0$ = per unit voltage corresponding to zero times rated exciting current, obtained by projecting the straight-line part of the excitation curve to the vertical axis. (See curve *B*, Fig. 2.)

As the voltage $E$ above saturation voltage is increased, the self-reactance of the winding approaches the reactance of an air-core reactor of the same dimensions. This reactance is always greater than the leakage reactance.

**Validity of the Use of Components in Calculations Involving Transformers.** When symmetrical or other components are used in solving unsymmetrical three-phase system problems, equivalent circuits are required to replace the transformer bank in each of the component networks. As the application of methods of components is based upon superpositions, they can be rigorously applied to transformers only if the parameters of the transformer are constant; however, they can be satisfactorily applied in many cases by assuming constant parameters, provided their variations would have negligible influence on calculations within the degree of precision required. In unsymmetrical short-circuit calculations, for example, exciting currents are usually neglected because they are a negligible part of the total short-circuit current. Neglecting exciting current is equivalent to assuming that the mutual impedance between windings and the self-impedances of windings are infinite relative to the other impedances of the system, and that the resultant calculated voltages across transformer windings will be below normal or not sufficiently above normal to invalidate this assumption. In general, if mutual impedances between transformer windings and the self-impedances of the windings can be assumed infinite relative to the other impedances of the system, a transformer bank can be replaced by equivalent circuits with exciting current neglected in the component networks, and a solution obtained by a method of components in which calculated values are within the usual required degree of precision.

Although a transformer bank of three identical single-phase units is physically a symmetrical circuit, it is not electrically symmetrical when the voltages across windings are badly unbalanced. Because of their non-linear character, the exciting impedances of the windings in this case are unequal and the circuit unsymmetrical in the three phases. However, the dissymmetry can be neglected where the unbalanced exciting impedances remain very large relative to other system impedances. In such cases, it makes little difference in calculations whether normal or infinite exciting impedance is used in the equivalent circuits. On the other hand, if transformer exciting impedances and other system impedances are of the same order of magnitude, the transformer bank cannot be even approximately represented by equivalent circuits of constant impedances in the component networks. The effects of saturation on magnetizing reactance must be taken into account as well as the fact that the transformer bank is an unsymmetrical circuit.

A method of including the effects of saturation on magnetizing reactance in an unsymmetrical circuit is given in Chapter V for the case of one or two open conductors in a circuit supplying ungrounded and unloaded transformers, where the capacitive reactances of the circuit and normal transformer magnetizing reactance are of the same order of magnitude.

**Balanced Three-Phase Operation.** The above discussion does not apply to balanced three-phase operation in which there are only positive-sequence quantities and superposition is not applied. A mutual impedance, corresponding to balanced voltages (see Problem 1); can be used in the positive-sequence equivalent circuit of Fig. 1 (*a*) for the transformer, regardless of its magnitude. This is done in power system calculations under normal operating conditions when a higher degree of precision is required than would be obtained by neglecting transformer exciting current; calculations of power loss and voltage regulation are examples.

## FUNDAMENTAL-FREQUENCY EQUIVALENT CIRCUITS

In developing equivalent circuits in this chapter to replace transformer and autotransformer circuits in the component networks, a physically symmetrical three-phase transformer circuit will be assumed to be electrically symmetrical also. This is equivalent to neglecting the effect of transformer saturation on exciting impedances. The validity of the equivalent circuits can be tested by determining resultant voltages across transformer windings by means of the equivalent circuits; if such resultant voltages do not appreciably change the magnitude of the assumed exciting impedance (including infinite assumed exciting impedance), the equivalent circuits will, in general, be adequate.

**Multiwindings.** Transformers with more than two windings are common. For example, there may be two generators connected to individual primary windings with a common secondary, or a single primary may serve two separate secondary circuits; a third (tertiary) winding may supply a synchronous condenser, or a third Δ-connected winding in a Y-Y-connected transformer bank may be used to suppress undesirable harmonic voltages. Transformers with four windings, although not so common as two- and three-winding transformers, are found in service; and five-winding transformers are a possibility.

**Open-Circuit and Short-Circuit Impedance Tests.** The impedances required in constructing equivalent circuits for multiwinding transformers can be obtained from open-circuit and short-circuit impedance tests. Open-circuit impedance tests give the open-circuit or self-impedances of the various windings with all other windings open. Short-circuit impedance tests give the short-circuit impedance between two windings with all other windings open. As the mutual impedance is very high relative to the leakage impedances (except at voltages above full saturation, not considered in this chapter), short-circuit impedance tests between windings give substantially the leakage impedances between windings taken two at a time with all other windings open.

**Per Unit Equivalent Circuits.** Unless otherwise specified, all equivalent circuits for transformers and autotransformers developed in this chapter are per unit or per cent equivalent circuits which are the same referred to all windings. In the equivalent circuit for a multiwinding transformer, base voltages in the various windings are in direct proportion to the number of turns in these windings. If base voltage is taken as rated voltage in a specified winding, base voltages in the other winding or windings are their open-circuit voltages with rated voltage applied to the specified winding. These base voltages will be called *base winding voltages.* The leakage impedances between windings are based on the same kva; if given impedances between windings are based on different rated kva's, they must be expressed on the same base kva before the equivalent circuit is constructed. (Per unit impedances vary directly as base kva and inversely as the square of base voltage.)

The per unit equivalent circuit given in Fig. 1 (*a*) of this chapter for a two-winding single-phase transformer with exciting current included was developed from equations based on the theory of electromagnetically coupled coils. The same equivalent circuit, given in Volume I, page 39, Fig. 14(*e*), was developed by setting up an assumed three-terminal equivalent circuit with three branch impedances to be evaluated. These branch impedances were evaluated by equating open-circuit and short-circuit impedances in the actual transformer to those in assumed equivalent circuit. Either of these methods, or a combination of the two methods, can be used to develop equivalent circuits. In any case, it is necessary that the equivalent circuit have the same number of terminals as the transformer has windings, with an additional terminal connected to zero-potential if exciting current is included; it must also have as many independent branch impedances as there are independent short-circuit impedances between windings taken two at a time with the other windings open.

## TRANSFORMER BANKS OF THREE IDENTICAL SINGLE-PHASE UNITS

In the following discussions of banks of two-, three-, four-, and five-winding transformers, it is assumed that windings are connected in Y or Δ. Scott-connected and zigzag-connected windings are discussed at the end of this chapter.

In the component networks of a three-phase system, base kva is kva per phase, and base voltages in the circuits at the transformer terminals are line-to-neutral voltages.

In developing equivalent circuits for banks of three identical single-phase transformers, the per unit equivalent circuit for the single-phase unit is first determined. Rated voltage in one winding is selected as base voltage for that winding; base voltages in the other windings are then determined from turn ratios, so that base voltages in windings are in direct proportion to the number of turns in the respective windings. Base kva is the same for all windings. The equivalent circuit for a single unit is also the equivalent circuit to replace the transformer bank in the positive- and negative-sequence networks. With Y-connected transformer windings, base line-to-neutral voltage in the three-phase circuit connected to the Y is the same as base winding voltage. With Δ-connected windings, base voltage in the Δ and base line-to-line voltage in three-phase circuit connected to the Δ are the same as base winding voltage; line-to-neutral voltage in the connected circuit is $1/\sqrt{3}$ times base winding voltage. The per unit impedance per phase of a Δ-connected circuit based on line-to-line voltage is the same as that of its equivalent Y based on the same kva and line-to-neutral voltage. Therefore, in determining currents and voltages *outside* the transformer bank, Δ-connected windings can be understood to be replaced by Y-connected windings of the same per unit impedances. Positive-sequence line currents and voltages to neutral in passing through a Y-Δ or Δ-Y transformer bank are shifted 90° in phase — whether forward or backward can be determined from the transformer connection diagram — whereas negative-sequence currents and voltages are shifted 90° in the direction opposite to the positive-sequence shift. (Volume I, pages 101–105.) Except for this shift in phase, positive- and negative-sequence equivalent circuits are independent of the manner of connecting the windings (Y or Δ); they are also independent of the method of grounding.

**Zero-Sequence Equivalent Circuits.** Although zero-sequence equivalent circuits depend upon the manner of connecting the windings (Y or Δ) and the method of grounding, they can be obtained from the per unit equivalent circuit for the single-phase unit by the following modifications:

1. The equivalent impedance of a Δ-connected winding is shorted to the zero-potential bus of the zero-sequence network; but there is no connection between the equivalent circuit and the circuit at the Δ terminals. See Figs. 3(*c*) and 4(*d*).
2. The connection to the circuit at the terminals of an ungrounded Y-connection winding is open. See Fig. 4(*f*).
3. An impedance $Z_n$ between the neutral of a Y and ground is expressed in per unit on system kva per phase and base line-to-neutral voltage in the Y-connected windings. $3Z_n$ in per unit is then added

to the impedance viewed from the Y terminal. See Figs. 3(*c*) and 4(*d*).

4. An impedance $Z_n$ in the corner of a Δ is expressed in per unit on system kva per phase and base line-to-line voltage in the Δ circuit. $Z_n/3$ in per unit is then added to the equivalent impedance of the Δ. See Figs. 3(*d*) and 4(*f*).

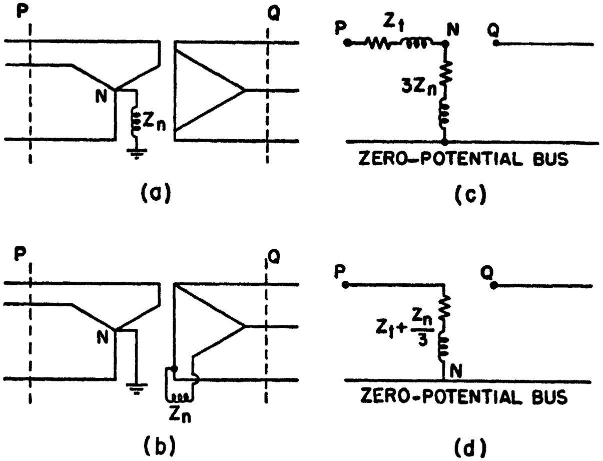


FIG. 3. Y-Δ bank with (*a*) neutral grounded through $Z_n$, (*b*) $Z_n$ in a corner of the Δ, and (*c*) and (*d*) zero-sequence equivalent circuits for (*a*) and (*b*), respectively, where $Z_t = Z_{ps}$ = leakage impedance between windings.

**Two-Winding Transformers.** Per unit equivalent circuits to replace the transformer are shown in Figs. 1(*a*) and 1(*b*), with exciting current included and neglected, respectively. These are also the equivalent circuits to replace the transformer bank of three single-phase units in the positive- and negative-sequence networks.

Zero-sequence equivalent circuits for two-winding transformer banks with exciting current neglected are shown in Fig. 18, page 102, Volume I, when the neutral of the Y-connected windings is solidly grounded or ungrounded. Impedances $Z_n$ in the neutral of the Y and in a corner of the Δ of a Δ-Y bank are shown in Figs. 3(*a*) and 3(*b*) of this chapter; their zero-sequence equivalent circuits with exciting currents neglected are given in Figs. 3(*c*) and 3(*d*), respectively. As the zero-sequence network is constructed on an equivalent per phase basis, $3Z_n$ is added to $Z_t$ with $Z_n$ in the neutral of the Y, and $Z_n/3$ is added to $Z_t$ with $Z_n$ in a corner of the Δ, where $Z_t$ is the per unit leakage impedance per phase of the bank. $3Z_n$ and $Z_n/3$ are expressed in per unit on the same kva per phase as $Z_t$; base voltage for $3Z_n$ is base line-to-neutral voltage of the Y-connected circuit; for $Z_n/3$ it is base line-to-line voltage in the Δ circuit. The following example illustrates the calculation of $Z_0$ for the two cases.

**Problem 2.** A transformer bank connected Y-Δ of three single-phase units each rated 1667 kva, 26.5 kv/46 kv Y- 4.6 kv, with 7 per cent reactance, has (*a*) 10 ohms reactance in the neutral of the Y; (*b*) 10 ohms reactance inserted in a corner of the Δ, Y solidly grounded. Determine the zero-sequence reactance $x_o$ viewed from the Y terminals of the bank in per cent on a three-phase kva base of 5000 kva (1667 kva per phase).

*Solution.* Base line-to-neutral voltage in the Y-connected circuit is 26.5 kv; in the Δ-connected circuit, base voltage is the line-to-line voltage 4.6 kv.

$$(a)\qquad 3Z_n = \frac{3 \times 10 \times 1667}{(26.5)^2 \times 10} = 7.1\%;\ x_o = (7 + 7.1)\% = 14.1\%$$

$$(b)\qquad \frac{Z_n}{3} = \frac{\frac{10}{3} \times 1667}{(4.6)^2 \times 10} = 26.2\%;\ x_o = (7 + 26.2)\% = 33.2\%$$

*Note*: The Y-Y bank and the Y-Y-Δ bank with the neutrals of the Y's grounded through a common impedance are discussed under autotransformers.

**Three-Winding Transformers.** The per unit equivalent circuit for a three-winding transformer with exciting current neglected, given in Fig. 16(*b*), page 42, Volume I, is repeated in Fig. 4(*a*) of this chapter. This circuit is also used to replace a transformer bank of three identical single-phase units in the positive- and negative-sequence networks. As a first approximation, the exciting current can be included in this equivalent circuit by connecting the per unit mutual impedance between point *J* and neutral *N*, as indicated in Fig. 4(*b*). In Figs. 4(*a*) and 4(*b*), subscripts *p*, *s*, and *t* indicate primary, secondary, and tertiary windings, respectively. $Z_p$, $Z_s$, and $Z_t$, which are the equivalent per unit leakage impedances of these windings, are determined as follows:

$$\begin{aligned} Z_p &= \tfrac{1}{2}(Z_{ps} + Z_{pt} - Z_{st}) \\ Z_s &= \tfrac{1}{2}(Z_{ps} + Z_{st} - Z_{pt}) \qquad [5] \\ Z_t &= \tfrac{1}{2}(Z_{pt} + Z_{st} - Z_{ps}) \end{aligned}$$

where $Z_{ps}$, $Z_{pt}$, and $Z_{st}$ are per unit short-circuit impedances of the two windings indicated by the subscripts with the third winding open, based on the same kva per phase and base winding voltages which are in direct proportion to their turns. The per unit mutual impedance at rated voltage can be determined as for the two-winding transformer. It is approximately equal to the per unit self-impedance of any winding less the average equivalent leakage impedance. For greater precision, the four-terminal equivalent circuit of Fig. 5(*a*) may be evaluated.

Zero-sequence equivalent circuits for banks of three-winding transformers with exciting current neglected are shown in Fig. 18, page 102, Volume I, for ungrounded or solidly grounded Y-connected windings. Figure 4(*c*) of this chapter shows a three-winding Y-Y-Δ transformer bank with the secondary windings connected in Y and grounded through an impedance $Z_n$; the zero-sequence equivalent circuit is shown in Fig. 4(*d*). This equivalent circuit is similar to the corresponding circuit with solidly grounded Y's except that $3Z_n$ is added to the equivalent impedance of the *Y*-connected windings which are grounded through $Z_n$. Expressed in per unit, $3Z_n$ is based on system

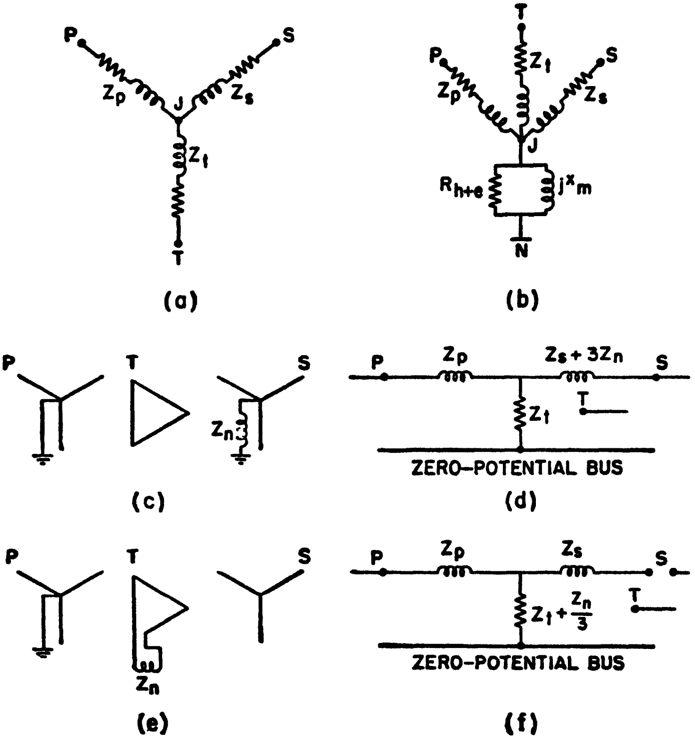


Fig. 4. Three-winding transformer banks. (*a*), (*b*) Positive-sequence equivalent circuits with exciting current neglected and included, respectively. (*c*) Y-Y-Δ bank with impedance $Z_n$ in neutral of Y. (*d*) Zero-sequence equivalent circuit for (*c*). (*e*) Y-Y-Δ bank with impedance $Z_n$ in corner of Δ. (*f*) Zero-sequence equivalent circuit for (*e*).

kva per phase and the base line-to-neutral voltage of the Y-connected circuit in which it is located. The voltage at the neutral of the Y is not given directly by Fig. 4(*d*) but can be determined by multiplying the zero-sequence current per phase at $S$ by $3Z_n$.

Figure 4(*e*) shows an impedance $Z_n$ inserted in a corner of the Δ tertiary of a Y-Y-Δ bank. As for the two-winding bank, this is equivalent to $\frac{1}{3}Z_n$ in each phase of the Δ and therefore $\frac{1}{3}Z_n$ is added to the equivalent impedance of the tertiary winding, as shown in Fig. 4(*f*), where $\frac{1}{3}Z_n$ is expressed in per unit on the system base kva per phase and base line-to-line voltage in the Δ.

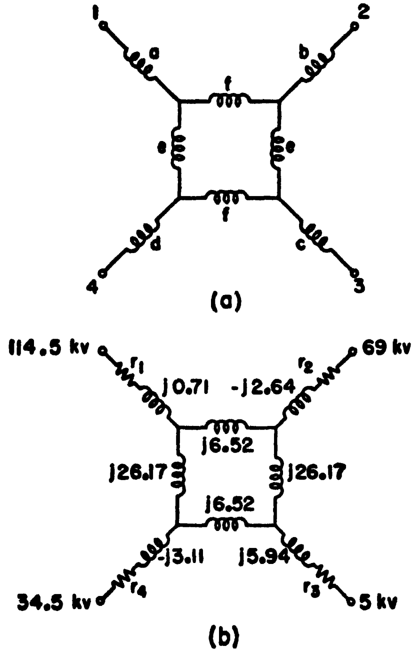


FIG. 5. (*a*) Assumed equivalent circuit for four-winding transformer. See [7]–[13] for evaluation of branch impedances. (*b*) Per cent equivalent circuit for transformer bank of Problem 3 with reactances in per cent based on 11,667 kva per phase.

**Four-Winding Transformers.** An equivalent circuit for a single-phase four-winding transformer with exciting current neglected must have four terminals and at least six independent impedances corresponding to the six short-circuit impedances between the four windings taken two at a time, with the other two windings open. If possible, the equivalent circuit should be free from internal negative impedance links, so that it can be used on either the d-c or the a-c calculating table. The equivalent circuit shown in Fig. 5(*a*) developed by F. M. Starr,[2] satisfies these conditions. The four terminals 1, 2, 3, 4 of Fig. 5(*a*) correspond to the four windings. In this circuit there are eight impedances, but only six of them are independent. The six branch impedances *a*, *b*, *c*, *d*, *e*, and *f* in the assumed equivalent circuit of Fig. 5(*a*) are to be evaluated in terms of the six measurable per unit short-circuit impedances between transformer windings. Let $Z$ with two subscripts represent these impedances, the subscripts referring to the two windings considered. Thus, $Z_{12}$ is the per unit short-circuit impedance between windings 1 and 2, with windings 3 and 4 open.

The transformer impedances and the branch impedances of Fig. 5(*a*) are related by the equations

$$\begin{aligned}
Z_{12} &= a + b + \frac{(2e + f)f}{2(e + f)} \\
Z_{13} &= a + c + \frac{(e + f)}{2} \\
Z_{14} &= a + d + \frac{(2f + e)e}{2(e + f)} \\
Z_{23} &= b + c + \frac{(2f + e)e}{2(e + f)} \\
Z_{24} &= b + d + \frac{(e + f)}{2} \\
Z_{34} &= c + d + \frac{(2e + f)f}{2(e + f)}
\end{aligned} \qquad [6]$$

If $a$, $b$, $c$, and $d$ are eliminated from these equations, there results

$$\begin{aligned}
(Z_{13} + Z_{24} - Z_{12} - Z_{34}) &= \frac{e^2}{(e + f)} = K_1 \\
(Z_{13} + Z_{24} - Z_{14} - Z_{23}) &= \frac{f^2}{(e + f)} = K_2
\end{aligned} \qquad [7]$$

From [7], $e$ and $f$ can be evaluated in terms of $K_1$ and $K_2$ which are determined from given transformer impedances.

$$e = \sqrt{K_2 K_1} + K_1 \qquad [8]$$

$$f = \sqrt{K_2 K_1} + K_2 \qquad [9]$$

The indicated roots in [8] and [9] have plus and minus values, but only the positive values of the roots will be considered here since it is desired to avoid negative values for $e$ and $f$. With $e$ and $f$ known, $a$, $b$, $c$, and $d$ from [6] are

$$a = \frac{Z_{12} + Z_{14} - Z_{24}}{2} - \frac{ef}{2(e + f)} \qquad [10]$$

$$b = \frac{Z_{12} + Z_{23} - Z_{13}}{2} - \frac{ef}{2(e + f)} \qquad [11]$$

$$c = \frac{Z_{23} + Z_{34} - Z_{24}}{2} - \frac{ef}{2(e + f)} \qquad [12]$$

$$d = \frac{Z_{34} + Z_{14} - Z_{13}}{2} - \frac{ef}{2(e + f)} \qquad [13]$$

It is desirable that links *e* and *f* be positive, which in turn requires that

$$\begin{aligned} Z_{13} + Z_{24} &> Z_{12} + Z_{34} \\ Z_{13} + Z_{24} &> Z_{14} + Z_{23} \end{aligned} \qquad [14]$$

These relations can always be satisfied if care is taken in assigning the numbers 1, 2, 3, and 4 to transformer windings. To illustrate, let the six impedances of a four-winding transformer, whose windings are $v$, $x$, $y$, and $z$, be grouped in three pairs as follows:

$$\begin{aligned} Z_{vx} + Z_{yz} &= A \\ Z_{vy} + Z_{xz} &= B \\ Z_{vz} + Z_{xy} &= C \end{aligned} \qquad [15]$$

One of the indicated sums must be larger than either of the other two. Suppose that $B$ is larger than either $A$ or $C$. Then, referring to equations [14], let $Z_{vy} + Z_{xz}$ correspond to $Z_{13} + Z_{24}$. This means that the letters $v$, $y$, $x$, and $z$ must correspond with the numbers 1, 3, 2, and 4, respectively, in order to satisfy equations [14].

The impedances as defined and used above include both resistance and reactance components. If resistance is to be included, a simpler procedure is to determine the equivalent circuit on the basis of leakage reactances only and then to add to the four radial links $a$, $b$, $c$, $d$ of Fig. 5(*a*) the resistances of their respective windings.

A four-winding bank of three single-phase units is replaced in the positive and negative-sequence networks by the equivalent circuit of Fig. 5(*a*) for a single-phase unit, windings connected in Δ being understood to be replaced by their equivalent Y's, as for the two- and three-winding banks. The same circuit is also used in the zero-sequence network with the modifications previously explained.

**Problem 3.** A transformer bank of three single-phase four-winding units is between circuits which have line-to-line voltages at no load of 114.5 kv, 69 kv, 34.5 kv, and 5 kv. The transformer windings connected to the circuits are designated arbitrarily as $v$, $x$, $y$, and $z$, respectively. The six measured per cent short-circuit reactances between windings, taken two at a time with the other windings open based on 11,667 kva per phase and no-load voltages are:

| Circuit Kilovolts (Line-to-Line) | Per Cent Reactance | Symbol |
|---|---|---|
| 114.5 to 69 | 3.95 | $Z_{vx}$ |
| 114.5 to 34.5 | 13.30 | $Z_{vy}$ |
| 114.5 to 5 | 23.00 | $Z_{vz}$ |
| 69.0 to 34.5 | 10.60 | $Z_{xy}$ |
| 69.0 to 5 | 19.00 | $Z_{xz}$ |
| 34.5 to 5 | 8.70 | $Z_{yz}$ |

Construct the per cent equivalent circuit with exciting current neglected for use in the positive-sequence network.

*Solution.* The connection of the windings (Y or Δ) is not given, nor is it required for the positive- or negative-sequence equivalent circuit when given reactances are based on the same kva per phase and base voltages in direct proportion to the number of turns.

By inspection, $Z_{vz} + Z_{xy}$ is greater than $Z_{vx} + Z_{yz}$ or $Z_{vy} + Z_{xz}$, which indicates that $v$, $z$, $x$, and $y$ should correspond to 1, 3, 2, and 4, respectively, in the equivalent circuit. If the above reactances are substituted in [7]–[13], the following values for the links of the equivalent circuit in per cent will be computed:

$$e = j26.17 \qquad a = j0.71 \qquad c = j5.94$$

$$f = j6.52 \qquad b = -j2.64 \qquad d = -j3.11$$

Note that all links except $b$ and $d$ are positive. Since these two negative reactances are small and appear in the radial links of the circuit, they may be added algebraically to reactances of transmission lines connected to the transformer, and negative links may thereby be avoided.

If it is desired to include resistance in the circuit, as previously indicated, the per cent resistances of the windings based on 11,667-kva per phase may be added to the four radial links as indicated in Fig. 5(*b*) by the symbols $r_1$, $r_2$, $r_3$, and $r_4$.

**Four- and Five-Terminal Equivalent Circuits.** The equivalent circuit for the four-winding transformer given above, and that for the five-winding transformer which follows, are not restricted in their application to transformers but can be used to represent other four- or five-terminal circuits.

**Five-Winding Transformers.** In a five-winding transformer there are ten measurable short-circuit impedances between windings, taken two at a time with the other windings open. An equivalent circuit for a five-winding transformer, with exciting current neglected, must therefore have five terminals and ten independent branch impedances. The equivalent circuit shown in Fig. 6, developed by L. C. Aicher, Jr.,[3] satisfies these conditions.

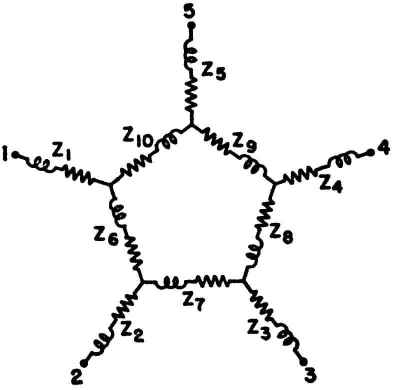


FIG. 6. Assumed equivalent circuit for five-winding transformer. See [17]–[19] for evaluation of branch impedances.

Let the five windings be numbered 1, 2, 3, 4, 5. The ten short-circuit impedances between windings, taken two at a time with the other windings open, will be designated by $Z$ with two subscripts indicating the two windings involved. The ten impedances are

$Z_{12}$, $Z_{13}$, $Z_{14}$, $Z_{15}$, $Z_{23}$, $Z_{24}$, $Z_{25}$, $Z_{34}$, $Z_{35}$, $Z_{45}$. In determining these impedances in per unit, base voltages in the various windings are in direct proportion to their turns, and all impedances are based on the same kva per phase.

In the equivalent circuit of Fig. 6, the five terminals marked 1, 2, 3, 4, 5 represent the corresponding terminals of the five windings. The radial links at the terminals are indicated by $Z$ with the subscripts of their terminals. The five impedances in the mesh are indicated by $Z_6$, $Z_7$, $Z_8$, $Z_9$, $Z_{10}$.

There are ten equations relating the ten short-circuit impedances of the transformer and the ten impedances of the equivalent circuit; the short-circuit impedances between transformer windings taken two at a time with the other windings open must equal the impedances in the equivalent circuit between the same terminals with all other terminals open.

Let $Z_t = Z_6 + Z_7 + Z_8 + Z_9 + Z_{10}$ = sum of the five mesh impedances. With this simplification, the ten equations relating impedances in transformers and equivalent circuit are

$$\begin{aligned}
Z_{12} &= Z_1 + Z_2 + \frac{Z_6}{Zt}(Z_7 + Z_8 + Z_9 + Z_{10}) \\
Z_{13} &= Z_1 + Z_3 + \frac{Z_6 + Z_7}{Z_t}(Z_8 + Z_9 + Z_{10}) \\
Z_{14} &= Z_1 + Z_4 + \frac{Z_9 + Z_{10}}{Z_t}(Z_6 + Z_7 + Z_8) \\
Z_{15} &= Z_1 + Z_5 + \frac{Z_{10}}{Z_t}(Z_6 + Z_7 + Z_8 + Z_9) \\
Z_{23} &= Z_2 + Z_3 + \frac{Z_7}{Zt}(Z_6 + Z_8 + Z_9 + Z_{10}) \\
Z_{24} &= Z_2 + Z_4 + \frac{Z_7 + Z_8}{Z_t}(Z_6 + Z_9 + Z_{10}) \\
Z_{25} &= Z_2 + Z_5 + \frac{Z_6 + Z_{10}}{Z_t}(Z_7 + Z_8 + Z_9) \\
Z_{34} &= Z_3 + Z_4 + \frac{Z_8}{Z_t}(Z_6 + Z_7 + Z_9 + Z_{10}) \\
Z_{35} &= Z_3 + Z_5 + \frac{Z_8 + Z_9}{Z_t}(Z_6 + Z_7 + Z_{10}) \\
Z_{45} &= Z_4 + Z_5 + \frac{Z_9}{Z_t}(Z_6 + Z_7 + Z_8 + Z_{10})
\end{aligned} \qquad [16]$$

where $Z_t = Z_6 + Z_7 + Z_8 + Z_9 + Z_{10}$.

Simultaneous solution of the above equations gives the ten impedances of the equivalent circuit in terms of the transformer short-circuit impedances. Solution is most simply performed by additions and subtractions of the ten initial equations of [16] to obtain ten new equations of which five contain none of the radial impedances ($Z_1$, $Z_2$, $Z_3$, $Z_4$, $Z_5$), and five contain only one of these impedances.

Let

$$
\begin{aligned}
L &= Z_{13} + Z_{24} - Z_{14} - Z_{23} = \frac{2Z_6Z_8}{Z_t} \\
M &= Z_{14} + Z_{25} - Z_{15} - Z_{24} = \frac{2Z_6Z_9}{Z_t} \\
P &= Z_{24} + Z_{35} - Z_{34} - Z_{25} = \frac{2Z_7Z_9}{Z_t} \\
Q &= Z_{13} + Z_{25} - Z_{12} - Z_{35} = \frac{2Z_7Z_{10}}{Z_t} \\
S &= Z_{14} + Z_{35} - Z_{13} - Z_{45} = \frac{2Z_8Z_{10}}{Z_t}
\end{aligned}
\qquad [17]
$$

In the five equations above, the radial impedances $Z_1$, $Z_2$, $Z_3$, $Z_4$, and $Z_5$ have been eliminated.

The five equations which contain only one of the radial impedances are obtained by adding two of the initial equations and subtracting a third.

Let

$$
\begin{aligned}
K &= Z_{12} + Z_{23} - Z_{13} = 2Z_2 + \frac{2Z_6Z_7}{Z_t} \\
N &= Z_{12} + Z_{15} - Z_{25} = 2Z_1 + \frac{2Z_6Z_{10}}{Z_t} \\
O &= Z_{23} + Z_{34} - Z_{24} = 2Z_3 + \frac{2Z_7Z_8}{Z_t} \\
R &= Z_{34} + Z_{45} - Z_{35} = 2Z_4 + \frac{2Z_8Z_9}{Z_t} \\
T &= Z_{15} + Z_{45} - Z_{14} = 2Z_5 + \frac{2Z_9Z_{10}}{Z_t}
\end{aligned}
\qquad [18]
$$

The radial impedances in the above equations can be determined when the mesh impedances are known.

The five mesh impedances can be expressed in terms of one of these impedances, for example, $Z_6$. From the first of equations [17],

$$
\begin{aligned}
Z_6 &= \frac{LZ_t}{2Z_8} = \frac{L}{2Z_8}(Z_6 + Z_7 + Z_8 + Z_9 + Z_{10}) \\
&= \frac{L}{2}\left(\frac{Z_6}{Z_8} + \frac{Z_7}{Z_8} + 1 + \frac{Z_9}{Z_8} + \frac{Z_{10}}{Z_8}\right)
\end{aligned}
$$

The ratios of the other mesh impedances to $Z_8$ can be determined from [17] in terms of the known values $L$, $M$, $P$, $Q$, and $S$, and substituted in the above equation to obtain $Z_6$. From [17], the other mesh impedances are expressed in terms of $Z_6$; and from [18], the radial impedances are expressed in terms $N$, $K$, $O$, $R$, $T$ and the mesh impedances:

$$
\begin{aligned}
Z_6 &= \frac{L}{2}\left(\frac{MQ}{PS} + \frac{Q}{S} + 1\right) + \frac{M}{2}\left(1 + \frac{Q}{P}\right) \\
Z_7 &= \frac{P}{M} Z_6 \\
Z_8 &= \frac{S}{Q} Z_7 = \frac{PS}{MQ} Z_6 \\
Z_9 &= \frac{M}{L} Z_8 = \frac{PS}{LQ} Z_6 \\
Z_{10} &= \frac{S}{L} Z_6 \\
Z_1 &= \frac{N}{2} - \frac{Z_6 Z_{10}}{Z_t} \\
Z_2 &= \frac{K}{2} - \frac{Z_6 Z_7}{Z_t} \\
Z_3 &= \frac{O}{2} - \frac{Z_7 Z_8}{Z_t} \\
Z_4 &= \frac{R}{2} - \frac{Z_8 Z_9}{Z_t} \\
Z_5 &= \frac{T}{2} - \frac{Z_9 Z_{10}}{Z_t}
\end{aligned}
\qquad [19]
$$

where $Z_t = Z_6 + Z_7 + Z_8 + Z_9 + Z_{10}$.

The impedances of the equivalent circuit of Fig. 6 are given in [19] in terms of constants determined from the transformer short-circuit impedances in the manner defined in [17] and [18]. The per unit equivalent circuit for the single-phase transformer is also the positive- or negative-sequence equivalent circuit; and the zero-sequence equivalent circuit can be obtained from it as previously explained and illustrated for two- and three-winding transformers.

## BANKS OF SINGLE-PHASE AUTOTRANSFORMERS

The autotransformer has a continuous winding, part of which is common to the two circuits connected to its terminals. See Fig. 7(*a*). The portion of the winding between $L$ and $N$ is common to both the high- and low-voltage circuits and is called the common winding; the portion of the winding between $H$ and $L$ is called the series winding; the total winding between $H$ and $N$ is the series-common winding. Expressed in amperes, the current in the series winding is the same as the current $I_H$ at the high-voltage terminal; the current in the common winding in amperes is the difference between the currents $I_L$ and $I_H$ at the low- and high-voltage terminals, respectively.

Autotransformers for three-phase systems may be banks of three single-phase units, or three-phase autotransformers. The most common connection of the windings is in Y; the neutral of the Y may be solidly grounded, grounded through impedance, or ungrounded. There is frequently a third winding connected in Δ, called the tertiary winding; more than three windings are occasionally used.

**Autotransformer Banks of Three Identical Single-Phase Units.** When there is no fault or dissymmetry *inside* the bank, equivalent circuits for autotransformer banks, as for transformer banks, are conveniently expressed in terms of impedances in per unit (or per cent), based on the same kva per phase and on base voltages in the windings which are directly proportional to the number of turns in the windings. Equivalent circuits with exciting current neglected will be developed for the autotransformer bank shown in Fig. 7(*b*) with windings connected in Y, the neutral grounded through an impedance $Z_N$, and a Δ-connected tertiary. These equivalent circuits will then be evaluated for the bank with solidly grounded neutral, with and without Δ tertiary; and for the bank with ungrounded neutral and a Δ tertiary.

Let the leakage impedances between windings be represented by the following symbols:

$Z_{sc-c}$ = impedance between series-common and common windings

$Z_{sc-s}$ = impedance between series-common and series windings

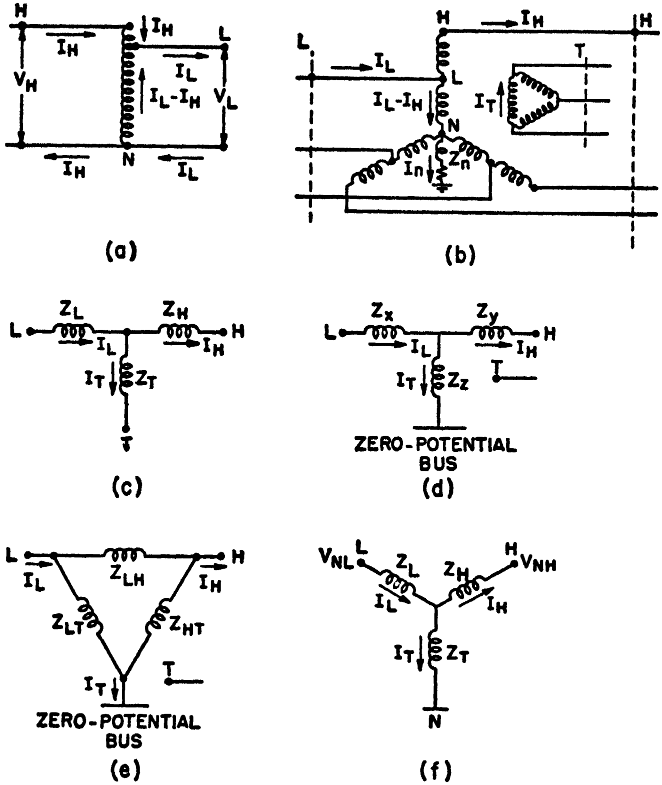


FIG. 7. (*a*) Single-phase autotransformer with indicated currents in amperes. (*b*) Autotransformer bank with Δ tertiary and neutral grounded through $Z_n$ with indicated currents in amperes. (*c*) Per unit equivalent circuit to replace (*b*) in the positive- or negative-sequence network, where $Z_L$, $Z_H$, and $Z_T$ are given by [20]. (*d*) Per unit equivalent circuit to replace (*b*) in the zero-sequence network where $Z_x$, $Z_y$, and $Z_z$ are to be evaluated. (See [32] for this evaluation.) (*e*) Per unit resonant Δ to replace an ungrounded autotransformer bank with Δ tertiary in the zero-sequence network, where $Z_{LH}$, $Z_{HT}$, and $Z_{LT}$ are to be evaluated. (See [39] for this evaluation.) (*f*) Auxiliary zero-sequence equivalent circuit for ungrounded autotransformer bank with Δ tertiary where voltages at $L$ and $H$ are referred to neutral.

$Z_{sc-t}$ = impedance between series-common and tertiary windings
$Z_{c-t}$ = impedance between common and tertiary windings
$Z_{c-s}$ = impedance between series and common windings
$Z_{s-t}$ = impedance between series and tertiary windings
$n_c, n_s, n_t$ = turns in common, series, and tertiary windings, respectively
$n_c + n_s$ = total turns in series-common winding

where the impedances are in per unit on the same kva base per phase, and base voltages in the windings are directly proportional to the number of turns in these windings. These impedances can be obtained by test in a manner similar to that used to obtain impedances between transformer windings. The per unit short-circuit impedance between any two windings is obtained with the third winding open and is the same referred to either winding.

The positive- or negative-sequence equivalent circuit, which is unaffected by the neutral connection, is indicated in Fig. 7(*c*). This equivalent circuit is similar to that of a three-winding transformer bank and is determined in a similar manner. (Volume I, page 42.) The impedances $Z_L$, $Z_H$, and $Z_T$ to be inserted in Fig. 7(*c*) to obtain the positive- or negative-sequence equivalent circuit are

$$\begin{aligned} Z_L &= \tfrac{1}{2}(Z_{sc-c} + Z_{c-t} - Z_{sc-t}) \\ Z_H &= \tfrac{1}{2}(Z_{sc-c} + Z_{sc-t} - Z_{c-t}) \qquad [20] \\ Z_T &= \tfrac{1}{2}(Z_{sc-t} + Z_{c-t} - Z_{sc-c}) \end{aligned}$$

It will be noted that the current in the common winding is not represented in Fig. 7(*c*). Its value in amperes is the difference between $I_L$ and $I_H$, when both are expressed in amperes.

If there is no Δ tertiary, or no circuit connected to the Δ tertiary, the point $T$ will be open-circuited in the positive- and negative-sequence networks, and the impedance between $L$ and $H$ becomes

$$Z_L + Z_H = Z_{sc-c}$$

The equivalent circuit is then similar to that of the two-winding transformer with exciting current neglected given in Fig. 1(*b*), with $Z_{sc-c}$ replacing $Z_{ps}$.

**Zero-Sequence Equivalent Circuits.** Figure 7(*d*) shows a zero-sequence equivalent circuit for the autotransformer bank of Fig. 7(*b*) in terms of $Z_x$, $Z_y$, and $Z_z$ which are to be evaluated. This equivalent circuit was first evaluated by Summers and McClure;[4] the evaluation given here is not that used in their development.

In Fig. 7(*d*), the terminal of $Z_z$ is shorted to the zero-potential bus; there is no connection between the equivalent circuit and the terminal $T$ of the Δ. The currents $I_L$, $I_H$, and $I_T$ are zero-sequence currents at terminals $H$ and $L$ and in the Δ, respectively, in per unit of base currents in these respective circuits. Arbitrarily assumed directions of current flow are indicated by arrows. Viewed from the terminals, the direction of current flow at $H$ and $L$ (whether toward the autotransformer and its equivalent circuit or toward the external circuits) will be the same in the actual as in the equivalent circuit. With exciting current neglected, the direction of $I_T$ will be such that the algebraic sum of the ampere-turns in each of the single-phase autotransformers is zero; in the equivalent circuit, the sum of the currents flowing towards a point (here the neutral of the equivalent Y) must be zero. In Fig. 7(*b*), the ampere-turns resulting from $I_L$ flowing from $L$ to $N$ is balanced by the ampere-turns resulting from $I_H$ flowing from $N$ to $H$ plus those resulting from $I_T$ in the same direction as $I_H$. Therefore in the per unit equivalent circuit of Fig. 7(*d*), $I_L = I_H + I_T$.

Voltages at $H$ and $L$ are per unit zero-sequence voltages referred to ground. Let

$V_{NH}$ and $V_{NL}$ = voltages to neutral at $H$ and $L$ in per unit of high and low base voltages, respectively

$I_N$, $V_N$ = current in the neutral and voltage at the neutral in per unit of base line current and base line-to-neutral voltage, respectively, in the *low-voltage* circuit

$Z_N = \dfrac{V_N}{I_N}$ = neutral impedance in per unit of base ohms in the *low-voltage* circuit

$Z_N$, $I_N$, and $V_N$ are here arbitrarily defined in terms of base quantities in the low-voltage circuit; they could as well have been defined in terms of base quantities in the high-voltage circuit.

Since the equivalent circuit has three terminals, three equations must be written relating the indicated unknown impedances $Z_x$, $Z_y$, and $Z_z$ of the proposed equivalent circuit to the known impedances in the actual circuit:

1. With the high-voltage terminals open in Fig. 7(*b*), the per unit zero-sequence impedance of the autotransformer bank viewed from the low-voltage terminals is $Z_{c-t} + 3Z_N$. With point $H$ open in the equivalent circuit of Fig. 7(*d*), the impedance viewed from $L$ is $Z_x + Z_z$. The equation relating the zero-sequence impedance of the equivalent circuit and that of the autotransformer bank for this condition is

$$Z_x + Z_z = Z_{c-t} + 3Z_N \qquad [21]$$

2. Similarly, with the low-voltage terminals open in Fig. 7(*b*) and the terminal $L$ open in Fig. 7(*d*), the zero-sequence impedance viewed from $H$ is

$$Z_y + Z_z = Z_{sc-t} + 3Z_N\left(\frac{n_c}{n_c + n_s}\right)^2 \qquad [22]$$

where the second term on the right-hand side of [22] gives the neutral impedance in per unit of high-voltage base impedance.

3. With the Δ tertiary open or removed, the current in per unit is the same at the high- and low-voltage terminals.

$$I_H = I_L = I \qquad [23]$$

where $I$ is the current in either winding in per unit of base current in that winding. The per unit impedance between $H$ and $L$ in the circuit of Fig. 7(*b*) with the Δ open is $(V_L - V_H)/I$. The per unit impedance in the equivalent circuit of Fig. 7(*d*) between $H$ and $L$ with $Z_z$ disconnected from the zero-potential bus and on open circuit is $Z_x + Z_y$. Therefore,

$$Z_x + Z_y = \frac{V_L - V_H}{I} \qquad [24]$$

where $(V_L - V_H)/I$ is to be evaluated.

With the Δ tertiary open or removed, the voltages at $H$ and $L$ referred to neutral are independent of the voltage above ground of the neutral. In per unit, with direction of current flow as indicated in Fig. 7(*b*) by arrows,

$$V_{NL} - V_{NH} = IZ_{sc-r} \qquad [25]$$

The neutral current $I_N$ in amperes is three times the difference between the currents at the low- and high-voltage terminals when both are expressed in amperes. In per unit of base current in the low-voltage circuit, with $I_H$ in per unit of base current in the high-voltage circuit multiplied by $n_c/(n_c + n_s)$ to refer it to base current in the low-voltage circuit,

$$I_N = 3\left[I_L - I_H\left(\frac{n_c}{n_c + n_s}\right)\right] \qquad [26]$$

Replacing $I_L$ and $I_H$ in [26] by $I$ from [23],

$$I_N = 3I\left(\frac{n_s}{n_c + n_s}\right) \qquad [27]$$

In per unit of base voltage in the low-voltage circuit,

$$V_N = I_N Z_N = 3IZ_N \frac{n_s}{n_c + n_s} \qquad [28]$$

$$V_L = V_N + V_{NL} = 3IZ_N \frac{n_s}{n_c + n_s} + V_{NL} \qquad [29]$$

In per unit of base voltage in the high-voltage circuit, with $V_N$ in [28] multiplied by $n_c/(n_c + n_s)$ to express it on this base voltage,

$$V_H = V_N \frac{n_c}{n_c + n_s} + V_{NH} = 3IZ_N \frac{n_c n_s}{(n_c + n_s)^2} + V_{NH} \qquad [30]$$

Substitution of [29] and [30] in [24], with $V_{NL} - V_{NH}$ replaced by $IZ_{sc-c}$ from [25], gives

$$Z_x + Z_y = \frac{V_L - V_H}{I} = Z_{sc-c} + 3Z_N \left(\frac{n_s}{n_c + n_s}\right)^2 \qquad [31]$$

Simultaneous solution of [21], [22], and [31] gives the values of $Z_x$, $Z_y$, and $Z_z$ to be substituted in the equivalent circuit of Fig. 7(*d*):

$$\begin{aligned} Z_x &= \tfrac{1}{2}(Z_{sc-c} + Z_{c-t} - Z_{sc-t}) + 3Z_N \frac{n_s}{n_c + n_s} \\ Z_y &= \tfrac{1}{2}(Z_{sc-c} + Z_{sc-t} - Z_{c-t}) - 3Z_N \frac{n_s n_c}{(n_c + n_s)^2} \qquad [32] \\ Z_z &= \tfrac{1}{2}(Z_{sc-t} + Z_{c-t} - Z_{sc-c}) + 3Z_N \frac{n_c}{n_c + n_s} \end{aligned}$$

If $Z_N = 0$, $Z_x$, $Z_y$, and $Z_z$ are the same as $Z_L$, $Z_H$, and $Z_T$, respectively, given by [20].

The equivalent circuit of Fig. 7(*d*) can be used to replace the autotransformer bank in the zero-sequence network for any unbalanced operating condition where exciting current can be neglected and the dissymmetry is outside the bank. From it, the zero-sequence currents and voltages at the terminals *L*, *H*, and in the Δ (zero-sequence voltage across Δ windings is zero) in per unit of base currents and voltages in the high, low, and tertiary circuits, respectively, can be obtained.

The neutral of the Y is not represented in the equivalent circuit; therefore the current flowing from the neutral and the voltage-to-ground of the neutral are not given directly. The zero-sequence current in the common winding is the difference between the zero-sequence currents at the low- and high-voltage terminals when both are expressed in amperes. The current flowing in the neutral grounding impedance is three times the zero-sequence current in amperes in the common winding. The voltage above ground of the neutral in volts is the product of the current flowing from the neutral to ground in amperes and the neutral impedance in ohms.

*With solidly grounded neutral*, $Z_N = 0$, and the zero-sequence equivalent circuits, with and without a Δ tertiary, are the same as the corresponding positive-sequence equivalent circuits, except that the equivalent impedance $Z_T$ of the Δ tertiary is shorted to the zero-potential bus in the zero-sequence network, whereas in the positive-sequence network it may be connected to a three-phase circuit or left open.

*With no Δ tertiary*, the impedance $Z_z$ in Fig. 7(*d*) is disconnected from the zero-potential bus, and the equivalent circuit for the autotransformer bank in the zero-sequence network reduces to the impedance between $L$ and $H$ given by [31].

*The* Y-Y *transformer bank and the* Y-Y-Δ *bank with neutrals grounded through a common impedance* $Z_N$ and exciting impedance neglected can be replaced in the zero-sequence network by equivalent circuits similar to those for the autotransformer bank with neutral grounded through $Z_N$, without and with a Δ tertiary, respectively. The per unit leakage impedance between the high- and low-voltage transformer windings will replace $Z_{sc-c}$ in [31] and [32], and the per unit impedances between the tertiary and the high- and low-voltage windings, respectively, will replace $Z_{sc-t}$ and $Z_{c-t}$ in [32].

*In the autotransformer bank with ungrounded neutral and Δ tertiary*, $Z_N = \infty$; and the three impedances $Z_x$, $Z_y$, and $Z_z$ of the equivalent Y of Fig. 7(*d*) given by [32] become infinite. A zero-sequence equivalent circuit for the autotransformer bank can be obtained by converting the equivalent Y to an equivalent Δ before equating $Z_N$ to $\infty$. A more direct procedure is to evaluate the elements of the equivalent Δ directly from the actual circuit. Figure 7(*e*) shows the assumed Δ with indicated impedances $Z_{LH}$, $Z_{LT}$, and $Z_{HT}$, which are to be evaluated.

Three equations are needed to relate the three impedances in the equivalent circuit to impedances in the autotransformer bank, corresponding to three assumed operating conditions:

1. With either the high- or low-voltage terminals of the autotransformer bank open and zero-sequence voltages applied to the other terminals, no current will flow. The impedance of the equivalent circuit for this condition must therefore be infinite, or

$$\frac{Z_{HT}(Z_{LH} + Z_{LT})}{Z_{HT} + Z_{LH} + Z_{LT}} = \infty \qquad [33]$$

Equation [33] is satisfied if

$$Z_{HT} + Z_{LH} + Z_{LT} = 0 \qquad [34]$$

With $Z_{LH} + Z_{HT} + Z_{LT} = 0$, the Δ cannot be replaced by an equivalent Y of finite impedances. An equivalent $\Delta$ in which the sum of the three branch impedances is zero is called a *resonant* $\Delta$.

2. If zero-sequence voltages are applied to the low-voltage terminals of the autotransformer bank with the high-voltage terminals shorted to ground, or vice versa, zero-sequence currents will flow in the three series windings and in the $\Delta$ tertiary, but no current will flow in the three common windings. The current flowing in the series and tertiary windings meets the impedance $Z_{s-t}$ which, as defined, is the same in per unit referred to either of these windings, base voltages in the windings being directly proportional to the number of turns in the windings. To refer $Z_{s-t}$ in per unit to the low-voltage winding it is multiplied by $(n_s/n_c)^2$; to refer it to the high-voltage winding it is multiplied by $[n_s/(n_c + n_s)]^2$.

With the high-voltage terminals shorted to ground, and zero-sequence voltage $V_L$ applied at the low-voltage terminals, in per unit,

$$\frac{V_L}{I_L} = Z_{s-t}\left(\frac{n_s}{n_c}\right)^2 \qquad [35]$$

In the equivalent circuit for this condition,

$$\frac{V_L}{I_L} = \frac{Z_{LH}Z_{LT}}{Z_{LH} + Z_{LT}} \qquad [36]$$

From [35] and [36],

$$\frac{Z_{LH}Z_{LT}}{Z_{LH} + Z_{LT}} = Z_{s-t}\left(\frac{n_s}{n_c}\right)^2 \qquad [37]$$

3. The corresponding equation, with the low-voltage terminals shorted and zero-sequence voltage $V_H$ applied to the high-voltage terminals, is

$$\frac{Z_{LH}Z_{HT}}{Z_{LH} + Z_{HT}} = Z_{s-t}\left(\frac{n_s}{n_c + n_s}\right)^2 \qquad [38]$$

Solution of the simultaneous equations [34], [37], and [38] gives

$$\begin{aligned} Z_{LH} &= Z_{s-t}\frac{n_s^2}{n_c(n_c + n_s)} \\ Z_{LT} &= -Z_{s-t}\frac{n_s}{n_c} \\ Z_{HT} &= Z_{s-t}\frac{n_s}{n_c + n_s} \end{aligned} \qquad [39]$$

Equations [39] give the impedances to be inserted in the equivalent $\Delta$ of Fig. 7(*e*). With the equivalent $\Delta$ replacing the autotransformer bank in the zero-sequence network, the three sequence networks can be set up on an a-c network analyzer and connected to satisfy given operating conditions; or, in analytic solutions, the impedances of the equivalent circuit can be combined with other system impedances in the usual manner to obtain the zero-sequence impedance of the system viewed from any point outside the autotransformer bank. With any one of the three terminals $H$, $L$, or $T$ open-circuited, the impedance of the equivalent circuit is infinite when exciting currents can be neglected.

The discussion given in the following chapter of open conductors in circuits supplying ungrounded transformer banks can be applied by analogy to circuits supplying ungrounded autotransformer banks.

The voltage of the ungrounded neutral above ground cannot be obtained directly from the equivalent circuit of Fig. 7(*e*). But in a system study, after the per unit zero-sequence currents and voltages at the autotransformer terminals $L$ and $H$ have been determined, the equivalent circuit of Fig. 7(*f*), with $Z_T$ shorted to neutral $N$ and $Z_H$, $Z_L$, and $Z_T$ defined by [20], can be used as an auxiliary circuit to determine the per unit zero-sequence voltages-to-neutral $V_{NL}$ and $V_{NH}$ at $L$ and $H$, respectively. Then, subtracting $V_{NL}$ from $V_L$ (or $V_{NH}$ from $V_H$), the voltage above ground of the neutral is obtained in per unit of base voltage in the low- (or high-) voltage circuit. Thus, from Fig. 7(*f*),

$$V_{NL} = I_T Z_T + I_L Z_L$$

$$I_T = I_L - I_H$$

and in per unit on the low-voltage base, the voltage above ground $V_N$ at the neutral is

$$V_N = V_L - V_{NL} = V_L - I_T Z_T - I_L Z_L \qquad [40]$$

If $I_T$ in [40] is replaced by $I_L - I_H$,

$$V_N = V_L - I_L(Z_T + Z_L) + I_H Z_T \qquad [41]$$

Equation [41] is independent of $I_T$; $I_L$ and $I_H$ are per unit currents in the directions indicated by arrows in Fig. 7(*f*); with either $I_L$ or $I_H$ in the reverse direction, the sign preceding $I_L$ or $I_H$ in [41] should be reversed.

**Relations between Autotransformer Impedances.** Although six impedances between windings have been listed and defined for the autotransformer bank with $\Delta$ tertiary, only three are independent when exciting current is neglected. The impedances usually given are

$Z_{sc-c}$, $Z_{sc-t}$, and $Z_{c-t}$; the other three impedances can be determined from them, if required.

The impedance $Z_{sc-c}$ can be determined by applying a voltage $V_H$ to the series-common winding with the common winding shorted as in Fig. 8(*a*) and measuring the current $I_H$ at the high-voltage terminal, the ratio $V_H/I_H$ giving the impedance $Z_{sc-c}$. If $V_H$ and $I_H$ are in volts and amperes, respectively, $Z_{sc-c}$ will be in ohms, referred to the series-common winding; if $V_H$ and $I_H$ are in per unit of base current and voltage, respectively, in the high-voltage circuit, $Z_{sc-c}$ will be in per unit. Since $Z_{sc-c}$ in per unit is the same referred to either winding, it could also be determined by applying a voltage $V_L$ to the common winding, with the series-common winding shorted, and measuring the current $I_L$ at $L$, the ratio $V_L/I_L$ giving $Z_{c-sc} = Z_{sc-c}$ in per unit.

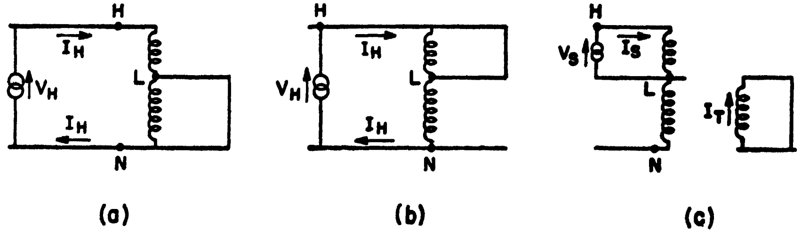


FIG. 8. Circuits for determining leakage impedances of autotransformer and relations between them.

From Fig. 8(*a*) it may be seen that, with the common winding shorted, the voltage $V_H$ in volts applied to the series-common winding is also the voltage $V_S$ across the series winding, and the current $I_H$ in amperes at the high-voltage terminal is also the current $I_S$ in the series winding. In Fig. 8(*a*) the ratio $V_H/I_H$, where $V_H$ and $I_H$ are in volts and amperes, respectively, can be used to determine the impedance $Z_{s-c}$ in ohms between the series and common windings referred to the series winding:

$$\frac{V_H \text{ (volts)}}{I_H \text{ (amp)}} = \frac{V_S \text{ (volts)}}{I_S \text{ (amp)}}$$

$$= Z_{sc-c} \text{ in ohms referred to series-common winding}$$

$$= Z_{s-c} \text{ in ohms referred to series winding}$$

In per unit on the same base kva per phase and with base voltages directly proportional to the number of turns in the respective windings,

$$Z_{s-c} = Z_{c-s} = Z_{sc-c}\left(\frac{n_c + n_s}{n_s}\right)^2 \qquad [42]$$

In Fig. 8(*b*), with $V_H$ applied to the series-common winding and the series winding shorted, the ratio $V_H/I_H$ where $V_H$ and $I_H$ are in volts and amperes, respectively, gives both $Z_{sc-s}$ and $Z_{c-s}$ in ohms. Therefore in per unit,

$$Z_{sc-s} = Z_{c-s}\left(\frac{n_c}{n_c + n_s}\right)^2 \qquad [43]$$

Substitution of [42] in [43] gives

$$Z_{sc-s} = Z_{sc-c}\left(\frac{n_c}{n_s}\right)^2 \qquad [44]$$

As the leakage impedance between any two windings is measured with all other windings open, the presence of a Δ tertiary will not affect [42]–[44].

The impedance $Z_{s-t}$, which is the only impedance required to construct the resonant Δ to replace the ungrounded autotransformer bank with Δ tertiary in the zero-sequence network, can be expressed in terms of the impedances $Z_{sc-c}$, $Z_{sc-t}$, and $Z_{c-t}$. With $V_S$ applied to the series windings and the tertiary winding shorted, as indicated in Fig. 8(*c*), and all quantities in per unit of their respective base quantities,

$$Z_{s-t} = \frac{V_S}{I_S} \qquad [45]$$

$$I_T = I_S \qquad [46]$$

In volts, $V_S = V_{NH} - V_{NL}$; and in amperes, $I_H = I_S$ and $I_L - I_H = 0$, since the current in the common winding is zero. These equations expressed in per unit are

$$V_S = V_{NH}\frac{n_c + n_s}{n_s} - V_{NL}\frac{n_c}{n_s} \qquad [47]$$

$$I_H = I_S\frac{n_c + n_s}{n_s} \qquad [48]$$

$$I_L = I_H\frac{n_c}{n_c + n_s} = I_S\frac{n_c}{n_s} \qquad [49]$$

From [48], [49], and [46],

$$I_H - I_L = I_S = I_T \qquad [50]$$

The equivalent circuit of Fig. 7(*f*), between *H*, *L*, and *N*, with $Z_T$ shorted to neutral *N* and the per unit impedances $Z_H$, $Z_L$, and $Z_T$

defined in [20], can be used to determine the voltages at $H$ and $L$ referred to neutral in terms of $Z_{sc-c}$, $Z_{sc-t}$, and $Z_{s-c}$. To satisfy [50], the directions of $I_H$ and $I_L$ in Fig. 7($f$) must be reversed. Then

$$V_{NH} = I_T Z_T + I_H Z_H \qquad [51]$$

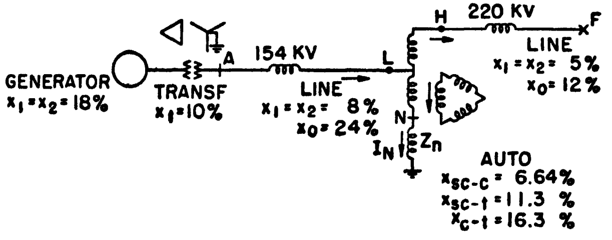
$$V_{NL} = I_T Z_T - I_L Z_L \qquad [52]$$

If $I_T$, $I_H$, and $I_L$ in [51] and [52] are replaced by their per unit values in terms of $I_S$ from [46], [48], and [49], respectively, and $Z_T$, $Z_H$, and $Z_L$ by their values given in [20], and if [51] and [52] are then substituted in [47], $V_S$ in terms of $I_S$ is obtained. Substitution of $V_S/I_S$ in [45] gives

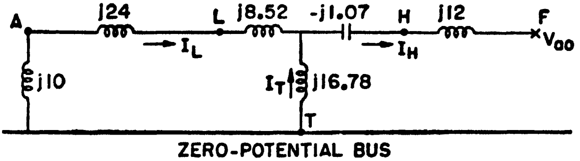
$$Z_{s-t} = Z_{sc-t}\frac{n_c + n_s}{n_s} - Z_{c-t}\frac{n_c}{n_s} + Z_{sc-c}\frac{n_c(n_c + n_s)}{n_s^2} \qquad [53]$$

Also, from [53] and [42],

$$Z_{sc-t} = Z_{s-t}\frac{n_s}{n_c + n_s} + Z_{c-t}\frac{n_c}{n_c + n_s} - Z_{c-s}\frac{n_c n_s}{(n_c + n_s)^2} \qquad [54]$$



(a)



(b)

FIG. 9. ($a$) One-line system diagram showing one phase of autotransformer bank with Δ tertiary and neutral grounded through $Z_n$, with line-to-ground fault at $F$. Per cent reactances are based on 20,000 kva per phase. ($b$) Per cent zero-sequence impedance diagram of ($a$) with $Z_n = j11.86$ ohms.

**Problem 4.** In the one-line system diagram of Fig. 9($a$), the autotransformer bank consists of three identical single-phase units. Base winding voltages in the common, series, and series-common windings are $154/\sqrt{3}$ kv, $66/\sqrt{3}$ kv, and $220/\sqrt{3}$ kv, respectively. The number of turns $n_c$, $n_s$, and $n_c + n_s$ are in the ratio 7 : 3 : 10, respectively. Resistances are not given. The reactances between windings, based on 60,000 kva (20,000 kva per phase) and base winding voltages are

$$x_{sc-c} = 6.64\%; \qquad x_{sc-t} = 11.3\%; \qquad x_{c-t} = 16.3\%$$

The Δ winding is rated 13.8 kv and 10,000 kva per phase. A line-to-ground fault occurs at $F$.

Draw the zero-sequence network of the system showing the autotransformer bank between $H$, $L$, and the zero-potential bus. Determine the zero-sequence voltages and currents at the high- and low-voltage terminals and at the neutral, and the current in the Δ tertiary.

(*a*) The neutral is grounded through a reactance of 11.86 ohms.

(*b*) The neutral is ungrounded.

*Solution.* (*a*) In per cent on the low-voltage common winding base,

$$3Z_N = j3\frac{11.86 \times 60{,}000}{(154)^2 \times 10} = j9.0\%$$

From [32] with resistance neglected,

$$Z_x = j\left[\tfrac{1}{2}(6.64 + 16.3 - 11.3) + 9\frac{3}{10}\right] = j8.52\%$$

$$Z_y = j\left[\tfrac{1}{2}(6.64 + 11.3 - 16.3) - 9\frac{3 \times 7}{(10)^2}\right] = -j1.07\%$$

$$Z_z = j\left[\tfrac{1}{2}(11.3 + 16.3 - 6.64) + 9\frac{7}{10}\right] = j16.78\%$$

By substituting these values of $Z_x$, $Z_y$, and $Z_z$ in Fig. 7(*d*), the equivalent circuit of the autotransformer bank is obtained. The zero-sequence network of the system is shown in Fig. 9(*b*) with impedances in per cent.

Viewed from the fault at $F$,

$$Z_0 = j\frac{42.52 \times 16.78}{59.3} - j1.07 + j12 = j22.95\%$$

$$Z_1 = Z_2 = j(18 + 10 + 8 + 6.64 + 5) = j47.64\%$$

$$Z_1 + Z_2 + Z_0 = j118.23\% = j1.182 \text{ per unit}$$

With unit voltage at $F$ before the fault, the components of per unit fault current at $F$ are

$$I_{a1} = I_{a2} = I_{a0} = \frac{1}{j1.182} = -j0.845$$

From the sequence networks, the symmetrical components of current and voltage are determined in the usual manner, and from them the phase currents and voltages.

The zero-sequence currents at $H$, $L$, and $N$ in the directions indicated by arrows in Fig. 9(*a*) are

$$I_H = I_{a0} = -j0.845 \text{ per unit}$$

$$= -j0.845 \times \frac{60{,}000}{\sqrt{3}\,220} = -j133 \text{ amp}$$

$$I_L = I_{a0} \times \frac{16.78}{59.3} = -j0.239 \text{ per unit}$$

$$= -j0.239 \times \frac{60{,}000}{\sqrt{3}\ 154} = -j53.7 \text{ amp}$$

$$I_N = 3(I_L - I_H) \text{ amp} = j238 \text{ amp}$$

$$V_N = I_N Z_N = j238 \times j11.86 \times 10^{-3} \text{ kv} = -2.82 \text{ kv}$$

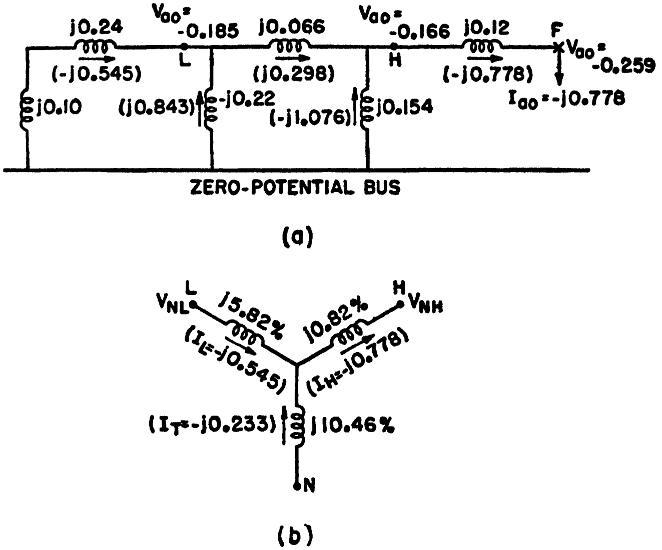


FIG. 10. (*a*) Per unit zero-sequence impedance diagram of Fig. 9(*a*) with $Z_n = \infty$. Per unit zero-sequence voltages at *L*, *H*, and *F* are indicated. Per unit zero-sequence currents are enclosed in parentheses with accompanying arrows indicating assumed directions. (*b*) Auxiliary zero-sequence per cent impedance diagram, with voltages at *H* and *L* referred to neutral.

The per unit zero-sequence current in the Δ tertiary in the direction indicated, based on 13.8 kv and 20,000 kva per phase, is

$$I_T = I_H - I_L = -j0.845 + j0.239 = -j0.606$$

Based on 13.8 kv and 10,000 kva per phase,

$$I_T = -j1.212 \text{ per unit} = -j1.212 \times \frac{10{,}000}{13.8} = -j878 \text{ amp}$$

(*b*) The reactance $x_{s-t}$ is the same in per cent referred to the series and tertiary windings, base voltages in these windings being $66/\sqrt{3}$ kv and 13.8 kv, respectively. In per cent based on 20,000 kva per phase, $Z_{s-t}$ calculated from [53] is

$$Z_{s-t} = j\left[11.3 \times \frac{220}{66} - 16.3\frac{154}{66} + 6.64\frac{154\ (220)}{(66)^2}\right] = j51.3\%$$

**Substitution of $Z_{s-t}$ in [39] gives the branch impedances of the equivalent circuit of Fig. 7(*e*) to replace the autotransformer in the zero-sequence network:**

$$Z_{LH} = j6.6\%; \qquad Z_{LT} = -j22.0\%; \qquad Z_{HT} = j15.4\%$$

The zero-sequence network of the system is shown in Fig. 10(*a*). Viewed from the fault in per cent,

$$Z_0 = j\frac{\left[\dfrac{34 \times (-22)}{12} + 6.6\right]15.4}{(-62.3 + 6.6) + 15.4} + j12 = j33.3\%$$

$Z_1$ and $Z_2$ viewed from the fault are unchanged by ungrounding the neutral of the Y; $Z_1 = Z_2 = 47.6\%$.

$$Z_1 + Z_2 + Z_0 = j33.3 + j2(47.6) = j128.5\%$$

With unit voltage at $F$ before the fault, the components of per unit fault current at $F$ are

$$I_{a1} = I_{a2} = I_{a0} = \frac{1}{j1.285} = -j0.778$$

In the per unit zero-sequence network of Fig. 10(*a*), the division of per unit current in the various branches and the zero-sequence voltages at $H$, $L$, and $F$ are indicated.

$$V_{a0} \text{ (at the fault)} = -I_{a0}Z_0 = j0.778 \times j0.333$$

$$= -0.259, \text{ based on } \frac{220}{\sqrt{3}} \text{ kv}$$

$$I_H = I_{a0} = -j0.778 \text{ per unit}$$

$$= -j0.778 \times \frac{60{,}000}{\sqrt{3}\,220} = -j122.5 \text{ amp}$$

$$V_H = j0.778 \times j0.213 = -0.166, \text{ based on } \frac{220}{\sqrt{3}} \text{ kv}$$

$$I_L = I_{a0} \times \frac{15.4}{-40.3} \times \frac{-22}{12} = -j0.545 \text{ per unit}$$

$$= -j0.545 \times \frac{60{,}000}{\sqrt{3}\,154} = -j122.5 \text{ amp}$$

$$V_L = j0.545 \times j0.34 = -0.185, \text{ based on } 154/\sqrt{3} \text{ kv}$$

Per unit $I_T$ can be obtained as the algebraic sum of the currents in the two branches of the resonant Δ connected to the zero potential bus; or from $I_L$ and $I_H$:

$$I_T = I_H - I_L = -j0.233 \text{ per unit}$$

$$= -j0.233 \times \frac{20{,}000}{13.8} = -j335 \text{ amp}$$

The current in the series winding in amperes is

$$I_S = I_H = -j122.5 \text{ amp}$$

Base current in the series winding is $60{,}000/(\sqrt{3}\,66) = 525$ amp

$$I_S = \frac{-j122.5}{525} = -j0.233 \text{ per unit}$$

As a check on numerical calculations, $I_S$ and $I_T$ are equal in per unit, and the current in the common winding $(I_L - I_H)$ in amperes is zero.

The per unit voltage at the neutral of the ungrounded Y of the autotransformer bank based on $154/\sqrt{3}$ kv can be obtained from [41] or from Fig. 10(*b*). $Z_L$, $Z_H$, and $Z_T$, calculated by substituting $Z_{sc-c}$, $Z_{sc-t}$, and $Z_{c-t}$ in [20] are $j5.82\%$, $j0.82\%$ and $j10.48\%$, respectively, and the zero-sequence currents and voltages at $L$ and $H$ are given above. From [41],

$$V_N = -0.185 - (-j0.545 \times j0.163) + (-j0.778 \times j0.1048) = -0.192 \text{ per unit}$$

$$= -0.192 \times \frac{154}{\sqrt{3}} = -17.1 \text{ kv}$$

**Exciting Currents in Banks of Single-Phase Autotransformers.** These currents have been neglected, and the mutual impedances between windings have been assumed to be infinite in the development of the positive-, negative-, and zero-sequence equivalent circuits given above. If exciting current is considered, the equivalent circuits for the autotransformer bank without a Δ tertiary will have three terminals; with a Δ tertiary, there will be four terminals. Magnetizing reactances are subject to saturation, and therefore autotransformer banks in which phase voltages across windings are badly unbalanced are unsymmetrical in the three phases and cannot be accurately represented by equivalent circuits in the sequence networks; however, they can be satisfactorily represented when mutual- and self-impedances of windings are large enough, relative to other system impedances involved in calculations, for their variations from constant values to have negligible influence on calculated values, within the required degree of precision. The discussion of overvoltages across transformer windings applies also to overvoltages across autotransformer windings.

## THREE-PHASE TRANSFORMERS AND AUTOTRANSFORMERS

**Transformers.** In three-phase transformers, the windings for the three phases are on separate legs which form part of the same iron core. There are two types of three-phase transformers in general use: the *core type* and the *shell type*. In the core-type transformer, the legs are in parallel and terminate in iron yokes. The usual core-type transformer has three legs, but there may be two additional outside legs of smaller cross section without windings, making a total of five legs. In the three-legged core-type transformer, there is a closed magnetic circuit for positive- and negative-sequence fluxes since their sum is zero, but the return path for zero-sequence fluxes is in air. A cross section of the iron core of a three-legged core-type transformer is shown in Fig. 11 (*a*). In the five-legged core-type transformer, there is an iron return path for zero-sequence fluxes through the legs without windings. In the shell-type transformer, the three legs carrying the phase windings are within the iron core. The windings of the middle phase are reversed in standard shell-type construction. A cross section of the iron core is shown in Fig. 11 (*b*). Because of the outside iron, there is a closed magnetic circuit for zero-sequence fluxes as well as for positive- and negative-sequence fluxes. With either type of three-phase transformer, there may be two, three, or more windings per phase.

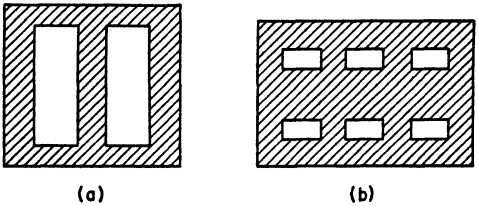


FIG. 11. Cross sections of iron cores of three-phase transformers. (*a*) Core type. (*b*) Shell type.

*Equivalent circuits* to replace three-phase transformers in the sequence networks are determined from open-circuit and short-circuit impedance tests in the same way as those for transformer banks of single-phase units. Open-circuit impedance tests give the self-impedances per phase of the various windings measured from their terminals with all other windings open; short-circuit impedance tests between two windings give the impedance per phase measured from the terminals of one winding with the second winding shorted. In three-phase transformers, the term winding includes the three phase windings. For the two-winding transformer, the two windings will be called high- and low-voltage windings, indicated by $H$ and $L$, respectively. In the three-winding transformer, the third winding will be called the tertiary winding, indicated by $T$. The voltage of the tertiary winding may be higher or lower than that of the high- or low-voltage winding. All impedances will be expressed in per cent or per unit based on the kva rating of the transformer and on voltages in the windings directly proportional to the number of turns in the windings.

**Positive- and Negative-Sequence Equivalent Circuits.** Since the positive- and negative-sequence flux paths for both types of three-phase transformers are principally in iron, the self-impedances of the windings are very high. When expressed in per unit or per cent, the open-circuit impedances for all windings are very nearly equal. Also, the short-circuit impedances between two windings are very nearly equal if measured from the terminals of either winding. Because the mutual reactance between windings is high relative to the leakage reactances, the short-circuit impedance between two windings is very nearly equal to the sum of the resistances and leakage reactances of the two windings. In the usual system problem, positive- or negative-sequence exciting currents can be neglected. The three-phase two-winding transformer is then replaced by a single impedance equal to the short-circuit impedance. Three- and four-winding transformers are replaced by three- and four-terminal equivalent circuits, respectively, similar to those for transformer banks of three single-phase units.

**Zero-Sequence Equivalent Circuits.** In the transformer bank of three single-phase units, the open-circuit and short-circuit zero-sequence impedances are the same as those of positive sequence if the paths for currents of both sequences are the same. This may not be the case in three-phase transformers.

The following discussion of zero-sequence open-circuit and short-circuit impedances of core-type and shell-type three-phase transformers, together with figures and typical values of impedances, is taken directly from a paper[5] by A. N. Garin. In this paper, a distinction is made between the complete zero-sequence equivalent circuit and the abbreviated equivalent circuit. The complete equivalent circuit is determined from open-circuit and short-circuit impedance tests in which paths are provided for zero-sequence currents in all windings. For example, the zero-sequence self- or open-circuit impedance of a Δ-connected winding can be obtained in test by impressing a voltage in a corner of the Δ with all other windings open, the ratio of one-third the impressed voltage to the Δ current giving the zero-sequence self-impedance per phase of the winding. With a second winding shorted, the short-circuit impedance measured from the Δ is obtained in a similar manner. The abbreviated equivalent circuit shows the transformer as seen from the system. Although insufficient for a complete solution of the transformer, it gives all the information needed for system calculations and, incidentally, all the information that can be obtained by testing the transformer without disturbing its internal connections. The complete zero-sequence equivalent circuit is determined from the open-circuit and short-circuit impedances in a manner similar to that used to determine the positive-sequence equivalent circuit for a two-winding transformer with exciting current included. The abbreviated equivalent circuit can be obtained from the complete equivalent circuit if connections to the system are left open for Δ- or ungrounded Y-connected windings and if terminals of equivalent impedances of Δ-connected windings are shorted to the zero-potential bus.

*Short-circuit zero-sequence impedances* for transformers of either core or shell type are practically as free from saturation as the positive-sequence short-circuit impedances, and the departures from symmetry are seldom large enough to be detected by the usual commercial test.

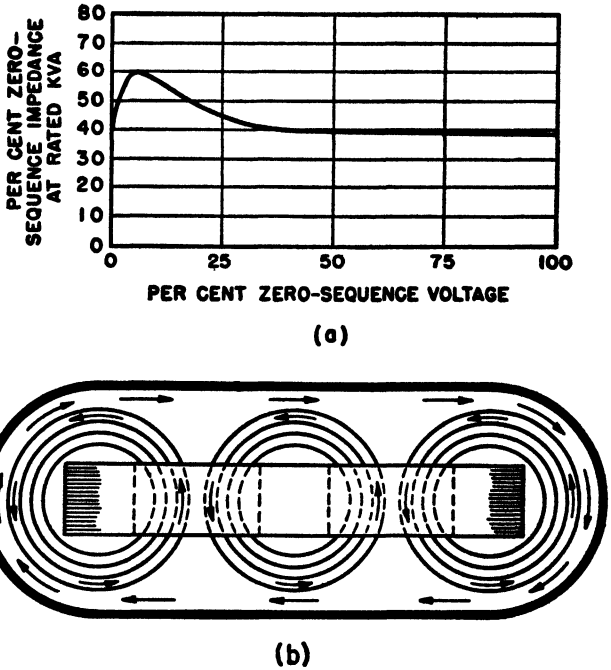


FIG. 12. (*a*) Typical open-circuit zero-sequence impedance curve for a three-phase, three-legged core-type transformer. (*b*) Tank of a three-phase, three-legged core-type transformer acting as a Δ-connected winding.

*Open-circuit zero-sequence impedances* of core-type and shell-type transformers are so different that they must be described separately.

In the three-legged core-type transformer, the positive-sequence flux paths are principally in iron while the sum of the zero-sequence fluxes in the three legs returns by air. The zero-sequence open-circuit impedance of any winding is therefore much lower and much less subject to saturation than the positive-sequence open-circuit impedance. The open-circuit zero-sequence impedance, as measured at the terminals of any winding, rarely exceeds 100%, and is more likely to be about 50%, based on the transformer rating. The variation of impedance with zero-sequence voltage is usually quite moderate, as shown in Fig. 12(*a*). The maximum value of impedance is found ordinarily at less than 5% zero-sequence voltage. The ratio of this maximum value to the final value, which is practically reached at 30% zero-sequence voltage, seldom exceeds 1.5 and may be less. The current in the three phases of a Y-connected winding or the voltage across the three phases of a Δ-connected winding (with an impressed voltage in a corner of the Δ) are usually equal to within 5%. During zero-sequence tests, no part of the core becomes saturated so long as zero-sequence voltage does not exceed 100%.

The transformer tank, as a rule, has no measurable influence on positive-sequence open-circuit impedance, but zero-sequence currents are similarly directed in the windings of the three phases and induce an oppositely directed current in the tank wall, so that in effect the tank acts as if it were a high-impedance Δ-connected winding. See Fig. 12(*b*). This Δ effect of the tank brings about a further reduction in the magnitude of the open-circuit zero-sequence impedance and a still closer balance of the currents of the three phases.

Thus, on open-circuit zero-sequence impedance test, a three-legged core-type transformer approaches quite closely the ideal symmetrical non-saturating circuit. During a line-to-ground fault, simultaneous excitation by voltages of several sequences, as determined by the system and the location of the fault, may result in saturation of parts of the core. The return path for the zero-sequence component of total flux, however, still lies outside the core iron, and the method of symmetrical components remains applicable, although in some cases its accuracy may be somewhat reduced.

A *shell-type core* provides a closed magnetic circuit both for positive- and for zero-sequence fluxes, but, except at very low excitation, the positive- and the zero-sequence open-circuit impedances are of a different order of magnitude. Open-circuit zero-sequence impedance of a shell-type transformer, as measured at the terminals of any winding, reaches its maximum value of several thousand per cent at about 20% zero-sequence voltage, Fig. 13(*a*), and then decreases very rapidly with increasing voltage, so that at 75% to 100% excitation it ranges between 500% and 100%, based on the rated kva of the transformer. This rapid decline is due to accelerated saturation of a part of core iron by the zero-sequence flux.

Owing to the reversal of windings of the middle phase, the flux density in the four shaded cross members of the core, Fig. 13(*b*), at any zero-sequence voltage is twice the density at the same positive-sequence voltage. For normal designs, zero-sequence voltage from 60% to 75% will result in full saturation of these parts of the core. The currents in the three phases of a Y-connected winding during the open-circuit zero-sequence impedance test are unbalanced, the current in the middle phase being 25% to 50% higher than the average value. This unbalance is also due principally to the reversal of the windings of the middle phase. Without a Δ-connected winding or a neutral tie (for example, a Y-Y transformer with the neutral of one Y only grounded), the zero-sequence equivalent circuit of the shell-type transformer would be an unsymmetrical circuit, subject to rapid saturation. However, on account of third-harmonic current requirements, shell-type transformers are seldom operated without a Δ-connected winding

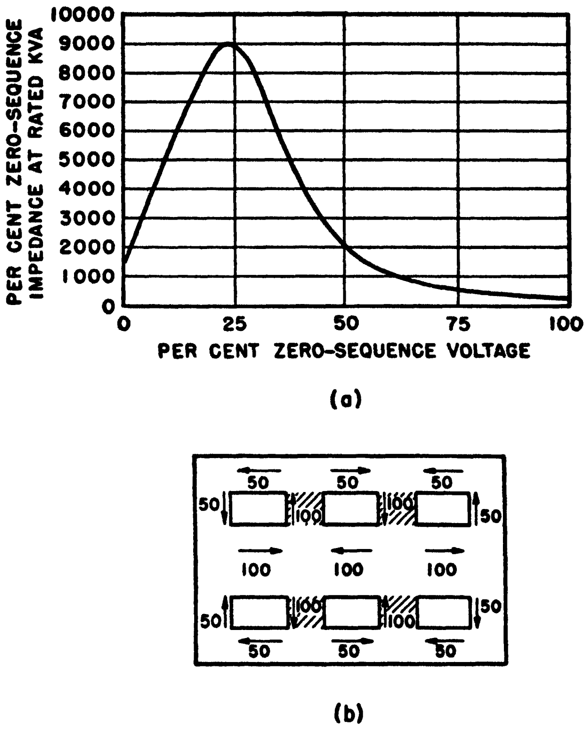


Fig. 13. (*a*) Typical open-circuit zero-sequence impedance curve for a three-phase shell-type transformer. (*b*) Saturation of core iron by zero-sequence flux in a three-phase shell-type transformer.

or a neutral tie. Any connection providing a path for third-harmonic currents will also provide a path for zero-sequence currents of operating frequency and will convert open-circuit zero-sequence impedances at terminals of other windings into short-circuit zero-sequence impedances, which, as stated above, satisfy the requirements of symmetry and are much less affected by saturation.

*Five-legged core-type transformers* have open-circuit zero-sequence characteristics intermediate between those of the shell type and the three-legged core type. As in shell-type transformers, the core provides a closed magnetic circuit for the zero-sequence flux, which in a normal design saturates at relatively low zero-sequence voltage. As in three-legged core-type transformers, the windings of the middle phase are not reversed and, at zero sequence voltages sufficient to saturate the outside legs, the Δ effect of the tank comes into play.

For core-type transformers the *resistance component* of short-circuit zero-sequence impedances is usually somewhat higher than the resistance component of corresponding positive-sequence impedances; but, in general, for both core-type and shell-type transformers the resistance components of corresponding short-circuit impedances of the two sequences are of the same order of magnitude. The resistance components of open-circuit zero-sequence impedances may be approximated by assuming the power factor of these impedances to be about 30% for core-type transformers and to vary from about 50% at low zero-sequence voltages to about 5% at high zero-sequence voltages for shell-type transformers. Thus, in system calculations zero-sequence impedances of transformers may usually be treated as pure reactances.

The value of zero-sequence resistance is not a reliable criterion of the ability of a transformer to carry zero-sequence currents of a specified duration without objectionable overheating, since it does not determine the magnitude of zero-sequence currents, nor does it give any information on the distribution of zero-sequence losses in the windings, the core, and the structural parts. The core and the structural parts may reach higher temperatures than the windings and are more likely to develop local hot spots. In practice, the magnitude and the distribution of zero-sequence losses, which are largely under the designer's control, are of importance only in exceptional cases when load currents contain a large zero-sequence component or when line-to-ground faults cannot be cleared within a few seconds.

**Zero-Sequence Equivalent Circuits for Two-Winding Three-Phase Transformers.** In the three-legged core-type transformer, since exciting current cannot be neglected, the complete equivalent circuit for a two-winding transformer will have three terminals $H$, $L$, and $M$, as in Fig. 14($a$), where the branch impedances $Z_H$, $Z_L$, and $Z_M$ are to be evaluated. Terminals $H$ and $L$ of the equivalent circuit correspond to the high- and low-voltage terminals, respectively, of one phase of the transformer. The neutral leads of the transformer are not represented in the equivalent circuit. The point $M$ is connected to the zero-potential bus of the network. The current in the impedance $Z_M$ of the equivalent circuit represents the zero-sequence exciting current, which is the difference between currents $I_H$ and $I_L$ and may be supplied by either winding.

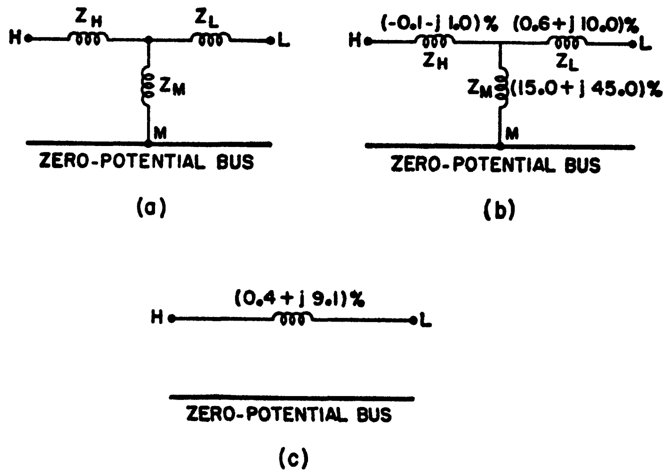


FIG. 14. Equivalent circuits for two-winding three-phase core-type transformer. ($a$) Complete equivalent circuit with impedances $Z_H$, $Z_L$, and $Z_M$ to be evaluated. ($b$) $Z_H$, $Z_L$, and $Z_M$ in ($a$) evaluated from zero-sequence impedance tests on a typical three-phase, three-legged core-type transformer. ($c$) Corresponding positive- or negative-sequence equivalent circuit in which $Z_M$ is relatively infinite.

Two open-circuit and two short-circuit impedance tests can be made, but only three of these impedances are required to determine the branch impedances $Z_H$, $Z_L$, and $Z_M$ of Fig. 14($a$). Typical zero-sequence open-circuit and short-circuit impedances of a three-phase, three-legged, core-type two-winding transformer with the usual concentric arrangement of windings determined by test are

$$(14.9 + j44.0)\% = \text{open-circuit impedance measured from } H$$
$$= Z_H + Z_M \text{ in Fig. } 14(a)$$

$$(15.6 + j55.0)\% = \text{open-circuit impedance measured from } L$$
$$= Z_L + Z_M \text{ in Fig. } 14(a)$$

$$(0.8 + j7.2)\% = \text{short-circuit impedance measured from } H$$
$$= Z_H + \frac{Z_L Z_M}{Z_L + Z_M} \text{ in Fig. 14}(a)$$
$$(0.5 + j9.0)\% = \text{short-circuit impedance measured from } L$$
$$= Z_L + \frac{Z_H Z_M}{Z_H + Z_M} \text{ in Fig. 14}(a)$$

The complete zero-sequence equivalent circuit is shown in Fig. 14(*b*). The positive-sequence equivalent circuit with exciting current neglected, which is a single impedance between *H* and *L* equal to the short-circuit impedance measured from either *H* or *L*, is indicated in Fig. 14(*c*).

It will be noted that the two open-circuit zero-sequence impedances have different values and that both of them are very much lower than the positive-sequence open-circuit impedance which may be of the order of 10,000%, corresponding to an exciting current of 1%. The two short-circuit impedances also differ from each other. In the equivalent circuit, $Z_H + Z_L$ is practically the same as the positive-sequence short-circuit impedance.

The complete zero-sequence equivalent circuit of Fig. 14(*a*) is used in the zero-sequence network when both windings are connected in Y and both are solidly grounded. If the high-voltage winding is connected in Δ, the terminal *H* in Fig. 14(*a*) is shorted to the zero-potential bus; if the low-voltage winding is connected in Δ, the terminal *L* is shorted to the zero-potential bus. In either case, the connection to the system from the Δ is open; likewise, for a winding connected in ungrounded Y, the connection to the system is open. For the Y-Δ connection of windings with the neutral of the Y grounded, the abbreviated equivalent circuit viewed from the Y terminal is a single impedance to the zero-potential bus. If the complete zero-sequence equivalent circuit of Fig. 14(*a*) with the terminal *L* (or *H*) shorted to the zero-potential bus is used, the current in the Δ and the exciting current in branch $Z_M$ of the equivalent circuit can be determined separately. Impedance in the neutral of a Y-connected winding or in the corner of a Δ-connected winding can be included in the equivalent circuits in the same way as for transformer banks of three single-phase units already described. See Figs. 4(*d*) and 4(*f*).

For two-winding transformers of the *shell type* or the *five-legged core type*, the value of $Z_M$ in Fig. 14(*a*) depends upon the magnitude of the voltages at the transformer terminals. At low voltages, $Z_M$ is sufficiently large to be omitted from the zero-sequence equivalent circuit just as mutual impedance is omitted from the positive-sequence equivalent circuit. On this basis, the approximate zero-sequence equivalent circuit for either a shell-type or a five-legged core-type transformer is a single impedance $Z_H + Z_L$ with two terminals $H$ and $L$. However, owing to saturation, the value of $Z_M$ decreases very rapidly with increasing excitation (see Fig. 13) and finally approaches values typical of three-legged core-type construction.

In a comparison of zero-sequence impedances of three-phase transformers with corresponding impedances of transformer banks of three single-phase units with the same connections of phase windings, the greatest difference is seen in Y-Y connected phase windings with the neutral of only one Y grounded. Viewed from the terminals of the grounded Y, the zero-sequence impedance of a transformer bank of three single-phase units is the same as the positive-sequence open-circuit impedance which is infinite relative to the positive-sequence short-circuit impedance, whereas that of the three-phase three-legged core-type transformer may be of the order of five times the positive-sequence short-circuit impedance; or, by special design, the mutual branch $Z_M$ may be made to approach zero or even reverse its sign. For this connection of phase windings, it is important to know whether the transformer is a bank of three single-phase units or a three-phase, three-legged, core-type transformer.

**Three-Winding Three-Phase Transformers.** The complete zero-sequence equivalent circuit of a three-winding transformer with windings $H$, $L$, and $T$ has four terminals, $H$, $L$, $T$, and $M$, with the terminal $M$ permanently connected to the zero-potential bus. The minimum number of branches is six, or the four-terminal eight-branch equivalent circuit discussed under the four-winding transformer may be used. If there is a Δ- or ungrounded Y-connected winding, the abbreviated zero-sequence equivalent circuit will have not more than three terminals. Consider a three-winding transformer with high, low, and tertiary windings indicated by $H$, $L$, and $T$ and connected Y-Y-Δ, respectively, with the neutrals of both Y's solidly grounded. The equivalent impedance $Z_T$ of the Δ will be permanently connected to the zero-potential bus as well as the terminal $M$, and the abbreviated equivalent circuit will have three terminals. Figures 15(*a*) and 15(*b*) show the positive- and zero-sequence equivalent circuits with the branch impedances $Z_H$, $Z_L$, $Z_T$, and $Z_{T\|M}$ to be evaluated. In the zero-sequence equivalent circuit, the impedance branch $Z_{T\|M}$ indicates that $Z_T$ as well as $Z_M$ is connected to the zero-potential bus; but there is no connection between the terminal $T$ of the Δ and the equivalent circuit.

The numerical values of branch impedances of the positive-sequence and of the abbreviated zero-sequence equivalent circuits for use in Fig. 15 may be quite different. As an example, tests on a 20,000-kva three-phase three-legged core-type transformer with concentric windings gave the following results:

| Positive-Sequence Equivalent Circuit | Abbreviated Zero-Sequence Equivalent Circuit |
|---|---|
| $Z_H = (0.24 + j9.20)\%$ | $Z_H = (0.70 + j6.14)\%$ |
| $Z_L = (0.09 - j1.25)\%$ | $Z_L = (-0.04 + j0.54)\%$ |
| $Z_T = (0.59 + j8.72)\%$ | $Z_{T\|M} = (0.84 + j5.20)\%$ |

If the $H$ (or $L$) winding is also connected in Δ, $Z_H$ (or $Z_L$) in Fig. 15(*b*) is also shorted to the zero-potential bus, with the terminal $H$ (or $L$) of the Δ disconnected from the equivalent circuit.

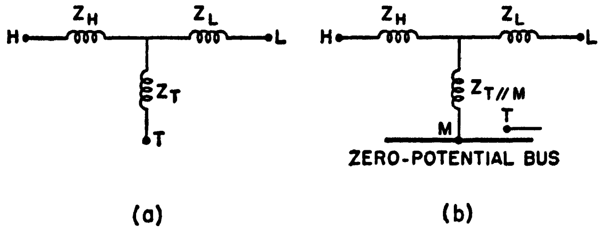


FIG. 15. Equivalent circuits for three-winding, three-phase transformers with Δ tertiary and branch impedances to be evaluated. (*a*) Positive-sequence. (*b*) Zero-sequence.

**Three-Phase Autotransformers.** An autotransformer can be subjected to the same zero-sequence impedance tests and represented by the same type of equivalent circuits as a straight transformer, but the numerical values of branch impedances in the equivalent circuits may be quite different.

The complete zero-sequence equivalent circuit of a three-phase autotransformer without a Δ-connected tertiary will be a three-terminal circuit as in the two-winding transformer shown in Fig. 14(*b*). For the three-phase autotransformer with a Δ-connected tertiary, the complete zero-sequence equivalent circuit has four terminals; the abbreviated zero-sequence equivalent circuit has three terminals, as in the three-winding transformer shown in Fig. 15(*b*).

## SCOTT-CONNECTED TRANSFORMER BANK WITH GROUNDED NEUTRAL

In Volume I, pages 353–356, the Scott-connected transformer bank with *ungrounded* neutral is discussed and its $\alpha$ and $\beta$ impedances determined. From these impedances, the positive- and negative-sequence self-impedances and the mutual impedance between the positive- and negative-sequence networks were obtained. There is no mutual impedance between the $\alpha$ and $\beta$ networks when two phases are symmetrical about the third phase; but the positive- and negative-sequence networks are mutually coupled because of dissymmetry in the phases.

With the neutral grounded, zero-sequence currents will flow during a ground fault on the three-phase side. When $\alpha\beta 0$ components are used, there will be mutual coupling between the $\alpha$ and the 0 networks but none with the $\beta$ network. When symmetrical components are used, there is coupling between all three networks. The zero-sequence self-impedance of the circuit is the same in either system of components.

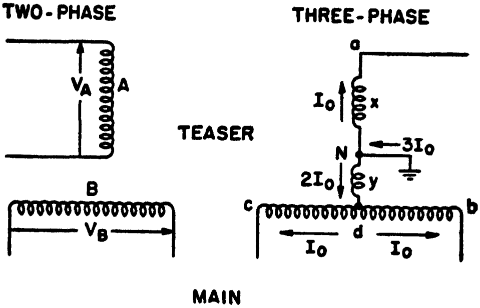


FIG. 16. Zero-sequence currents in Scott-connected transformer bank with grounded neutral.

The connection diagram of a Scott-connected transformer with grounded neutral is given in Fig. 16. *A* and *B* are the two-phase windings; transformer *B-bc* is the main transformer, and transformer *A-ad* is the teaser. Winding voltages of *bc*, *ad*, *aN*, and *Nd* are $\sqrt{3}$, $\frac{3}{2}$, 1, and $\frac{1}{2}$ times line-to-neutral voltage, respectively. When the neutral is ungrounded and exciting current is neglected, three leakage impedances only are required to determine $\alpha$ and $\beta$ self-impedances, or positive- and negative-sequence self-impedances and the mutual impedance between the positive- and negative-sequence networks. These leakage impedances are

$Z_{mt}$ = per unit leakage impedance of main transformer based on its rating (line-to-line voltage and $\frac{3}{2}$ rated kva per phase on three-phase side)

$Z_{tt}$ = per unit leakage impedance of teaser based on its rating ($\frac{3}{2}$ line-to-neutral voltage and $\frac{3}{2}$ rated kva per phase on three-phase side)

$Z_{it}$ = interlacing impedance — per unit impedance of windings *dc* and *db* of the main winding in parallel (with terminals *b* and *c* together) based on rated teaser voltage and current

When the neutral is grounded and a ground fault occurs on the three-phase side, $I_0$ flows in the section *Na* of the teaser winding *ad* and $2I_0$ in the section *Nd*. Let these sections be $x$ and $y$, respectively, as indicated in Fig. 16. With exciting current neglected and a 2 : 1 turn ratio, the zero-sequence ampere turns in windings $x$ and $y$ balance each other without producing flux through the complete core but only through the leakage paths; no voltage is therefore induced in winding *A* on the two-phase side by zero-sequence currents on the three-phase side.

Let

$Z_{xy}$ = per unit leakage impedance between windings $x$ and $y$ (sections *Na* and *Nd*) of the teaser winding *ad*, based on rated kva per phase on the three-phase side and base winding voltages which are rated line-to-neutral voltage in winding $x$ (*Na*) and one-half this voltage in winding $y$ (*Nd*)

With exciting current neglected, $Z_{xy}$ is the per unit impedance of either winding with the other winding short-circuited. As the turn ratio is 2 : 1, zero-sequence current in $y$ will be twice that in $x$ when expressed in amperes or in per unit on the same current base. In determining the impedances of the Scott-connected transformer to zero-sequence currents, transformer *aN–Nd* of the teaser and the interlacing impedance $Z_{it}$ will be considered separately and then combined.

The total impedance of the transformer *aN–Nd* to zero-sequence currents can be determined by connecting terminals *a* and *d* and applying a voltage to ground $V_0$ at the common point. As only zero-sequence current will flow, the impedance per phase is three times the total impedance. By superposition, the total current in $x$ and $y$ is the sum of the currents when $V_0$ is applied first to *a* with *d* grounded and then to *d* with *a* grounded. Let $V_0$ be expressed in per unit of rated line-to-neutral voltage. With $V_0$ applied at *a* and *d* grounded, per unit currents in $x$ and $y$ are equal, but base current in $y$ is twice rated line current. In per unit of their respective base currents,

$$I_x = I_y = \frac{V_0}{Z_{xy}}$$

In per unit of base line current,

$$I_x = \frac{V_0}{Z_{xy}} \quad \text{and} \quad I_y = \frac{2V_0}{Z_{xy}}$$

When $V_0$ is applied at *d* with *a* grounded, the applied voltage in per unit of base voltage in *y* is $2V_0$. In per unit of their respective base currents,

$$I_x = I_y = \frac{2V_0}{Z_{xy}}$$

In per unit of base line current,

$$I_x = \frac{2V_0}{Z_{xy}} \quad \text{and} \quad I_y = \frac{4V_0}{Z_{xy}}$$

The total current $3I_0$ in per unit of base line current when $V_0$ is applied to both *a* and *d* is

$$3I_0 = I_x + I_y = \frac{9V_0}{Z_{xy}} \quad \text{or} \quad \frac{V_0}{3I_0} = \tfrac{1}{9}Z_{xy} \qquad [55]$$

$$Z_{00} = \frac{V_0}{I_0} = \tfrac{1}{3}Z_{xy} \qquad [56]$$

The interlacing impedance, as defined, is based on $\frac{3}{2}$ rated kva per phase and $\frac{3}{2}$ line-to-neutral voltage; it must therefore be multiplied by $\frac{2}{3} \times (\frac{3}{2})^2 = \frac{3}{2}$ to be based on kva per phase and line-to-neutral voltage. $\frac{3}{2}Z_{it}$ is the parallel impedance of phases *b* and *c* to zero-sequence currents; there is no corresponding impedance in phase *a*.

$$Z_{a0} = 0 \quad Z_{b0} = 3Z_{it} \quad Z_{c0} = 3Z_{it}$$

In Volume I, page 229, the sequence self- and mutual impedances of an unsymmetrical circuit are expressed in terms of the sequence impedances of the phases. Thus,

$$\begin{aligned} Z_{00} &= \tfrac{1}{3}(Z_{a0} + Z_{b0} + Z_{c0}) = \tfrac{1}{3}(0 + 3Z_{it} + 3Z_{it}) = 2Z_{it} \\ Z_{10} &= \tfrac{1}{3}(Z_{a0} + aZ_{b0} + a^2Z_{c0}) = \tfrac{1}{3}(0 - 3Z_{it}) = -Z_{it} \qquad [57] \\ Z_{20} &= \tfrac{1}{3}(Z_{a0} + a^2Z_{b0} + aZ_{c0}) = \tfrac{1}{3}(0 - 3Z_{it}) = -Z_{it} \end{aligned}$$

$Z_{00}$ in [57] is the zero-sequence self-impedance; $Z_{10}$ is the ratio of positive-sequence voltage drop in the circuit produced by $I_{a0}$ to $I_{a0}$; $Z_{20}$ is the ratio of the negative-sequence voltage drop produced by $I_{a0}$ to $I_{a0}$. In a static circuit, $Z_{10} = Z_{20}$ and $Z_{20} = Z_{01}$.

When $Z_{00}$ in [56] is added to $Z_{00}$ in [57], the zero-sequence self-impedance of the Scott-connected transformer is obtained:

$$Z_{00} = \tfrac{1}{3}Z_{xy} + 2Z_{it} \qquad [58]$$

The positive-, negative-, and zero-sequence self-impedances and the mutual impedances between the sequence networks are as follows, where $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$ are given in Volume I, page 356:

$$Z_{11} = Z_{22} = \tfrac{1}{2}(Z_{tt} + Z_{it} + Z_{mt}) \qquad [59]$$

$$Z_{12} = Z_{21} = \tfrac{1}{2}(Z_{tt} + Z_{it} - Z_{mt}) \qquad [60]$$

$$Z_{00} = \tfrac{1}{3}Z_{xy} + 2Z_{it} \qquad [61]$$

$$Z_{10} = Z_{20} = Z_{01} = Z_{02} = -Z_{it} \qquad [62]$$

The $\alpha$, $\beta$, and 0 self-impedances and the mutual impedances between the $\alpha$ and 0 networks can be obtained from the sequence self- and mutual impedance, if [33]–[37] of Chapter I are used:

$$Z_{\alpha\alpha} = Z_{tt} + Z_{it} \qquad [63]$$

$$Z_{\beta\beta} = Z_{mt} \qquad [64]$$

$$Z_{00} = \tfrac{1}{3}Z_{xy} + 2Z_{it} \qquad [65]$$

$$Z_{\alpha\beta} = Z_{\beta\alpha} = Z_{\beta 0} = \beta_{0\beta} = 0 \qquad [66]$$

$$Z_{\alpha 0} = 2Z_{0\alpha} = -2Z_{it} \qquad [67]$$

## Grounding Transformers and Autotransformers

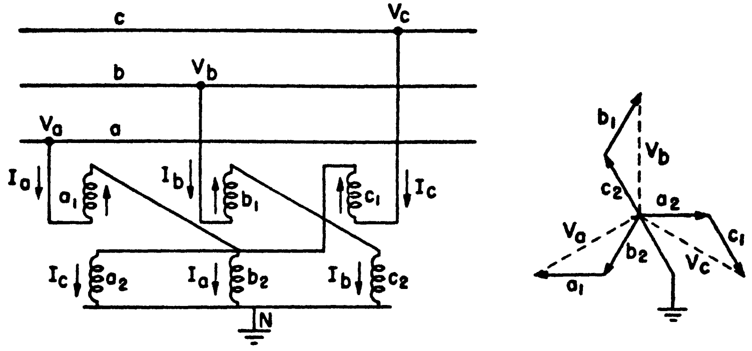
Grounding transformers or autotransformers are frequently used in three-phase systems to obtain a neutral for grounding purposes in a circuit which is otherwise ungrounded or insufficiently grounded. When used solely for grounding, they do not transmit power but serve only to provide a path for zero-sequence currents and thereby reduce system voltages during ground faults.

Two types of connections of the windings of grounding units will be considered: Y-Δ connections and zigzag connections, with the neutral of the Y and of the zigzag solidly grounded. Because of their relatively small sizes, grounding units are usually three-phase rather than banks of three single-phase units. The zigzag-connected three-phase unit is classified as an autotransformer because windings on the same leg of the iron core are connected in different phases. The three-phase Y-Δ unit may be similarly classified when it transmits no power and the terminals of the Δ-connected secondaries are not brought outside the unit.

**Δ-Y Connections.** With the Δ isolated, the impedance per phase to positive- and negative-sequence currents is the self-impedance of the Y which can usually be considered infinite relative to other system impedances. The zero-sequence impedance $Z_0$ per phase is the leakage impedance $Z_{ps}$ per phase between primary and secondary windings. Based on rated current per phase and rated line-to-neutral voltage,

$$Z_0 = Z_{ps} \qquad [68]$$

**Zigzag Connections.** The zigzag autotransformer is a two-winding unit in which the turn ratio is 1 : 1, and the primary and secondary windings are interconnected. This is indicated in the connection diagram of Fig. 17($a$) where $a_1$, $b_1$, and $c_1$ are the primary windings and $a_2$, $b_2$, and $c_2$ are their respective secondaries. The impedance of the autotransformer to positive- and negative-sequence currents is its exciting impedance which can usually be considered infinite relative to other system impedances. The normal voltage vector diagram is given in Fig. 17($b$), with voltages across windings on the same leg of the iron core shown as parallel lines. The normal voltage across each winding is $1/\sqrt{3}$ times normal line-to-neutral circuit voltage, or $\frac{1}{3}$ the normal line-to-line voltage.



(a) (b)

FIG. 17. Zigzag grounding transformer. ($a$) Wiring diagram. ($b$) Normal voltage vector diagram.

When a ground fault occurs, zero-sequence currents will flow in the autotransformer. Being equal and in phase, they will flow in opposite directions in corresponding primary and secondary windings, and therefore meet the leakage impedance between them. See Fig. 17($a$). The zero-sequence current in each phase flows through two coils, one primary and one secondary; under the assumption that leakage flux is similarly divided between all primaries and secondaries, the zero-sequence impedance per phase is the leakage impedance between one primary and its secondary.

The leakage impedance per phase of a zigzag autotransformer $Z_z$ is customarily given in ohms; if given in per cent based on specified current and voltage, it can be converted to ohms and then expressed in per unit of base ohms used in system calculations.

*Rating of Grounding Units.* The voltage rating of a grounding transformer or autotransformer is determined from the voltage of the circuit in which it is to be placed, but the current rating has no direct relation to the load current in the circuit. Grounding units are usually given two current ratings: (1) a nominal or continuous current rating based on the maximum current they could carry continuously; and (2) a short-time or short-circuit current rating based on the maximum value of the current they may be required to carry for a specified length of time. Unless otherwise specified, when only one current rating is given for an installed grounding transformer, it is understood to be the continuous or nominal current rating.

The continuous current rating is determined by the designer from the short-time current rating, the required reactance, and specifications in regard to duration of short-circuit duty and permissible temperature rise. The short-time current rating for a proposed grounding unit and its required reactance in ohms can be calculated, as illustrated in Problem 5.

## Selection of Grounding Units for Specified System Locations

When a line-to-ground short-circuit occurs on a circuit without a permanent ground, the voltages to ground of the unfaulted phases become $\sqrt{3}$ times their normal values. Grounding transformers are used to reduce these voltages to permissible values. The reduction in voltage is a function of the ratio of the zero-sequence reactance $x_0$ to the positive-sequence reactance $x_1$ and is influenced by resistance and capacitance. The lower this ratio $x_0/x_1$, the greater the reduction in voltage. However, a low reactance grounding unit has a higher current rating and is therefore more costly than one of the same type of higher reactance which may be adequate.

In Volume I, pages 178–181, charts are given of the voltages of the unfaulted phases in per unit of their normal values versus $x_0/x_1$, with $r_0/x_1$ as parameter, for various ratios of $r_1/x_1$ with negative-sequence impedance assumed equal to positive. If resistance and capacitance can be neglected and if positive- and negative-sequence impedance viewed from the fault are assumed to be equal, equations can be derived which relate the voltages to ground of the unfaulted phases at the fault and the ratio of $x_0/x_1$.

In the usual circuit, maximum zero-sequence current $I_0$ flows in the grounding transformer when a line-to-ground short-circuit occurs at its terminals. Let the fault be on phase *a* at *F*, the transformer terminals. Let $x_1$ and $x_0$ be positive- and zero-sequence reactance, respectively, viewed from the fault in per unit based on a chosen base kva per phase and normal line-to-neutral voltage at *F*. Then, with all quantities in per unit,

$$I_{a1} = I_{a2} = I_0 = -j\frac{1}{2x_1 + x_0} \qquad [69]$$

$$V_{a0} = -I_{a0}Z_0 = -\frac{x_0}{2x_1 + x_0}; \quad V_{a2} = -\frac{x_1}{2x_1 + x_0} \qquad [70]$$

$$V_{a1} = -(V_{a0} + V_{a2}) = \frac{x_1 + x_0}{2x_1 + x_0} \qquad [71]$$

$$|V_b| = |V_c| = \left| -\tfrac{1}{2}(V_{a1} + V_{a2}) - j\frac{\sqrt{3}}{2}(V_{a1} - V_{a2}) + V_{a0} \right|$$

$$= \left| \tfrac{3}{2}V_{a0} - j\frac{\sqrt{3}}{2} \right| = \sqrt{\tfrac{9}{4}|V_{a0}|^2 + \tfrac{3}{4}}$$

$$= \sqrt{\frac{9}{4}\left(\frac{x_0}{2x_1 + x_0}\right)^2 + \frac{3}{4}} \qquad [72]$$

$$|V_{a0}| = \tfrac{2}{3}\sqrt{|V_b|^2 - \tfrac{3}{4}} \qquad [73]$$

Also, $|V_{a0}| = \dfrac{x_0}{2x_1 + x_0}$, from which

$$x_0 = \frac{2x_1|V_{a0}|}{1 - |V_{a0}|} \qquad [74]$$

When $x_1$ has been determined and the zero-sequence reactance $x_0$ viewed from the fault is known, $I_0$ and $|V_b| = |V_c|$ can be calculated from [69] and [72]. When the permissible voltage $|V_b|$ or $|V_c|$ is given, $V_{a0}$ is calculated from [73] and substituted in [74] to give $x_0$ in terms of $x_1$. Equations [69]–[74] are system equations which apply to both Y-Δ- and zigzag-connected grounding units. Their application is illustrated in Problem 5.

**Problem 5.** It is proposed to install a grounding unit in a 69-kv circuit which is at present ungrounded. Positive-sequence reactance viewed from the proposed location has been calculated as 20% based on a three-phase kva of 20,000 and 69-kv line-to-line voltage. (*a*) What value of zero-sequence reactance $x_0$ in per unit is required to limit line-to-ground voltages on the unfaulted phases to 1.30 times their normal value of $69/\sqrt{3}$ kv? (*b*) What will be the short-time current rating of the grounding unit? and (*c*) its reactance per phase in ohms?

*Solution.* (*a*) From [73] and [74], in per unit based on 20,000/3 kva per phase and $69/\sqrt{3}$ line-to-neutral voltage,

$$|V_{a0}| = \tfrac{2}{3}\sqrt{(1.30)^2 - 0.75} = 0.646$$

$$x_0 = \frac{2 \times 0.20 \times 0.646}{1 - 0.646} = 0.73 \text{ per unit}$$

(*b*) From [69],

$$I_{a1} = I_{a2} = I_{a0} = -j\frac{1.0}{2(0.20) + 0.73} = 0.885$$

The current in the grounding unit in amperes is

$$I_m = I_{a0} = 0.885 \times \frac{20{,}000}{\sqrt{3} \times 69} = 148 \text{ amp}$$

The short-time current rating of the grounding unit is 148 amp.

(*c*) In ohms, the reactance per phase of the grounding unit $x_0$ is

$$x_0 = \frac{0.73 \times (69)^2 \times 10^3}{20{,}000} = 174 \text{ ohms}$$

## Δ-Zigzag and Y-Zigzag Transformers

The zigzag connection with a Δ or a Y as primary or secondary is used in the transmission of power. In Fig. 18(*a*), the wiring diagram for the zigzag connection is given, and the windings $a_1$, $b_1$, and $c_1$ of the Δ or Y are shown. The connection diagrams of the Δ-zigzag and Y-zigzag transformers are given in Figs. 18(*b*) and 18(*c*), respectively, with windings on the same core shown as parallel lines; these diagrams are also the no-load voltage vector diagrams. At no load, $V_a$, the line-to-neutral voltage at the terminals of the zigzag, is in phase with $V'_A$, the line-to-neutral voltage at the terminals of the Δ. See Fig. 18(*b*). In Fig. 18(*c*), $V_a$ at the terminals of the zigzag lags $V_A$ of the Y by 90°.

There is no shift in phase of voltages and currents in passing through a Δ-zigzag transformer bank. For this reason, Δ-zigzag banks are used to parallel existing Δ-Δ banks where a neutral is required for grounding. They are also used with synchronous converters, where the neutral of the zigzag is connected directly to the neutral wire of the Edison three-wire d-c system.

The shifts in phase of positive- and negative-sequence currents and voltages in passing through a Y-zigzag transformer bank are the same as those through a Y-Δ bank. The Y-zigzag bank will therefore operate in parallel with a Y-Δ bank.

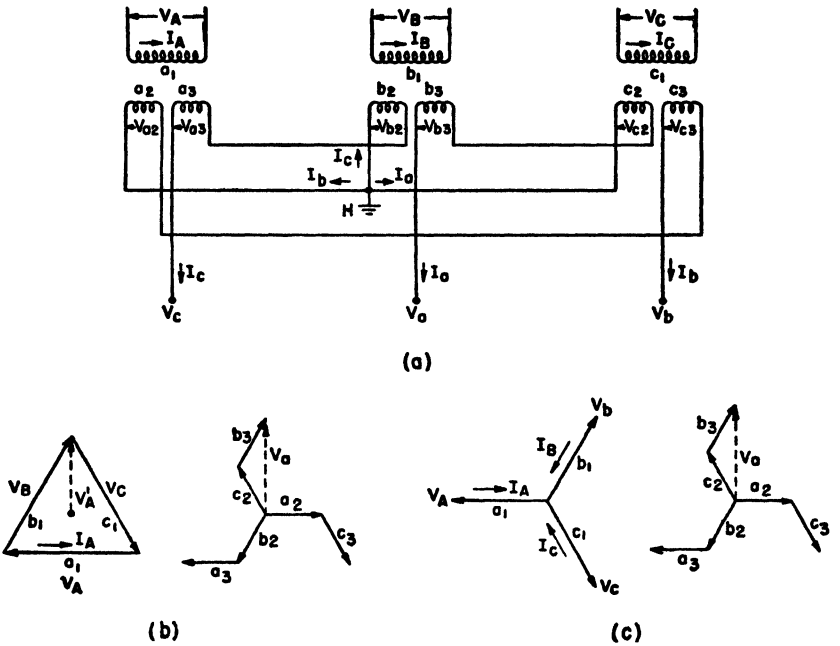


FIG. 18. Δ-zigzag and Y-zigzag transformer banks. (*a*) Wiring diagram of zigzag connections with windings of Δ or Y indicated, but not connected. (*b*), (*c*) Normal voltage vector diagrams of Δ-zigzag and Y-zigzag banks, respectively.

Δ-zigzag and Y-zigzag transformers will be analyzed as three-winding transformers with exciting currents neglected. The equivalent leakage impedances of any three windings $a_1$, $a_2$, and $a_3$ on the same core are the three short-circuit impedances between windings taken two at a time with the third winding open. Let

$Z_{12}$ = leakage impedance between windings $a_1$ and $a_2$

$Z_{13}$ = leakage impedance between windings $a_1$ and $a_3$

$Z_{23}$ = leakage impedance between windings $a_2$ and $a_3$

where all impedances are in per unit based on rated kva per phase and rated winding voltages which are in direct proportion to the number of turns. Base voltage in winding $a_1$ of the Δ is a line-to-line voltage;

in the Y it is a line-to-neutral voltage; in windings $a_2$ and $a_3$ of the zigzag which are assumed to have the same number of turns, it is $1/\sqrt{3}$ times base line-to-neutral voltage at the terminals of the zigzag.

The equivalent leakage impedances $Z_x$, $Z_y$, and $Z_z$ of windings $a_1$, $a_2$, and $a_3$ can be obtained from [5] for the three-winding transformer if $Z_x$, $Z_y$, and $Z_z$ replace $Z_p$, $Z_s$, and $Z_t$ and if $Z_{12}$, $Z_{13}$, and $Z_{23}$ replace $Z_{ps}$, $Z_{pt}$, and $Z_{st}$, respectively. Then,

$$Z_x = \tfrac{1}{2}(Z_{12} + Z_{13} - Z_{23}) \qquad [75]$$

$$Z_y = \tfrac{1}{2}(Z_{12} + Z_{23} - Z_{13}) \qquad [76]$$

$$Z_z = \tfrac{1}{2}(Z_{13} + Z_{23} - Z_{12}) \qquad [77]$$

The identical equivalent circuit for the three phases are shown in Figs. 19(*a*), (*b*), and (*c*) with the voltages and currents at their terminals indicated. The notation in Fig. 19 corresponds to that of Fig. 18(*a*). In Fig. 18(*a*), the assumed direction of voltage rise and of current flow in the windings are indicated by arrows, and currents and voltages are represented by symbols.

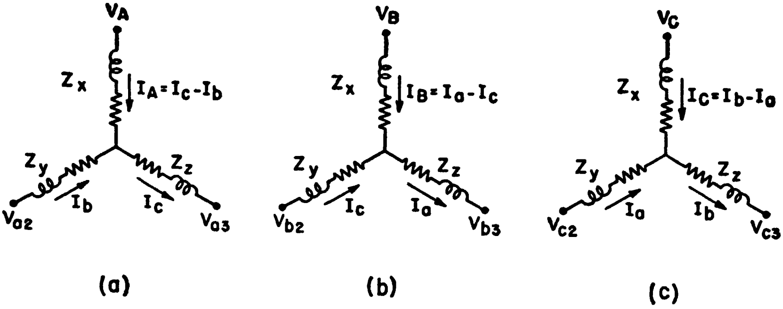


FIG. 19. (*a*), (*b*), (*c*) Identical equivalent circuits to replace each of the three-winding transformers, with currents and voltages indicated.

From Fig. 18(*a*),

$$V_a = V_{b3} - V_{c2} \qquad [78]$$

From Figs. 19(*b*) and 19(*c*),

$$V_{b3} = V_B - (I_a - I_c)Z_x - I_aZ_z \qquad [79]$$

$$V_{c2} = V_C - (I_b - I_a)Z_x + I_aZ_y \qquad [80]$$

Substitution of [79] and [80] in [78] gives

$$V_a = V_B - V_C - I_a(2Z_x + Z_y + Z_z) + (I_b + I_c)Z_x \qquad [81]$$

The transformer is symmetrical in the three phases; if it supplies a symmetrical circuit and if $V_A$, $V_B$, and $V_C$ are taken as positive-sequence applied voltages, $V_B - V_C = -j\sqrt{3}V_{A1}$, $V_a = V_{a1}$, $I_a = I_{a1}$, $I_b + I_c = -I_{a1}$, and [81] becomes

$$V_{a1} = -j\sqrt{3}\ V_{A1} - I_{a1}(3Z_x + Z_y + Z_z) \qquad [82]$$

If the applied voltages are of zero sequence, $V_B - V_C = 0$, $V_A = V_{a0}$, $I_a = I_{a0}$, $I_b + I_c = 2I_{a0}$, and [81] becomes

$$V_{a0} = -I_{a0}(Z_y + Z_z) \qquad [83]$$

In [82] and [83], $V_{a1}$ and $V_{a0}$ are the positive- and zero-sequence voltages, respectively, at the zigzag terminals in per unit of a base voltage equal to $1/\sqrt{3}$ times rated line-to-neutral voltage; $I_{a1}$ and $I_{a0}$ are in per unit of a base current equal to rated kva per phase divided by $1/\sqrt{3}$ times line-to-neutral kv.

Let $V'_{a1}$ and $V'_{a0}$ be the positive- and zero-sequence voltages at the zigzag terminals in per unit of line-to-neutral voltage, and $I'_{a1}$ and $I'_{a0}$ the corresponding currents in per unit of rated kva per phase divided by line-to-neutral kv. Then

$$V_{a1} = \sqrt{3}\ V'_{a1}; \quad V_{a0} = \sqrt{3}\ V'_{a0}; \quad I_{a1} = \frac{I'_{a1}}{\sqrt{3}}; \quad I_{a0} = \frac{I'_{a0}}{\sqrt{3}}$$

If these substitutions are made in [82] and [83] and both sides of the equations are divided by $\sqrt{3}$, there results

$$V'_{a1} = -jV_{A1} - I'_{a1}\frac{3Z_x + Z_y + Z_z}{3} \qquad [84]$$

$$V'_{a0} = -I'_{a0}\left(\frac{Z_y + Z_z}{3}\right) = -I'_{a0}\frac{Z_{23}}{3} \qquad [85]$$

From [84] at no load, $V'_{a1} = -jV_{A1}$. At no load the voltage-to-neutral at the terminals of the zigzag lags $V_A$ of the Δ or Y [see Figs. 18(*b*) and 18(*c*)] by 90°.

The per unit positive-sequence impedance $Z_1$ which is the same referred to either side of the transformer bank is

$$Z_1 = Z_x + \frac{Z_y + Z_z}{3} \qquad [86]$$

If $Z_x$, $Z_y$, and $Z_z$ are replaced by their values given in [75]–[77],

$$Z_1 = \tfrac{1}{2}(Z_{12} + Z_{13} - \tfrac{1}{3}Z_{23}) \qquad [87]$$

The per unit zero-sequence impedance of the bank viewed from the terminals of the zigzag, based on rated kva per phase and rated line-to-neutral voltage at the zigzag terminals is

$$Z_0 = \frac{Z_{23}}{3} \qquad [88]$$

where $Z_{23}$ is the per unit leakage impedance between any two windings of the zigzag on the same core, based on rated kva per phase and rated winding voltage.

Zero-sequence currents flowing in opposite direction in windings of the zigzag on the same core cause no resultant flux in the core, and therefore induce no voltage in the windings of the Δ or Y. The zero-sequence impedance viewed from the terminals of the Δ or ungrounded Y is infinite; if the Y is grounded, it is the self-impedance of the Y.

**Problem 6.** In a Δ-zigzag transformer bank, rated 7500 kva, rated voltage of the Δ is 13.8 kv, that of the zigzag windings is 23 kv. Line-to-line voltages at the zigzag terminals is 69 kv; line-to-neutral voltage is $69/\sqrt{3}$ kv. Per cent leakage reactance between a Δ winding and each of the zigzag windings on the same core is 12% based on 2500 kva per phase and rated winding voltages; leakage impedance between two zigzag windings on the same core is 9% based on 1250 kva and rated winding voltages, or 18% based on 2500 kva. Determine the impedances $Z_1$ and $Z_0$ for use in the positive-sequence and zero-sequence networks of the system, neglecting resistance and exciting currents.

*Solution.* Let Δ windings be $a_1$ and zigzag windings $a_2$ and $a_3$ as in Fig. 18($a$). Based on 2500 kva per phase and rated winding voltages,

$$x_{12} = x_{13} = 12\%$$

$$x_{23} = 18\%$$

$x_1$, based on 2500 kva per phase and line-to-neutral voltage at either set of terminals of the bank, from [87], is

$$x_1 = \tfrac{1}{2}(x_{12} + x_{13} - \tfrac{1}{3}x_{23}) = 9\%$$

From [88], $Z_0$ viewed from the zigzag terminals based on 2500 kva per phase and a line-to-neutral voltage of $69/\sqrt{3}$ is

$$x_0 = \frac{Z_{23}}{3} = \frac{18}{3} = 6\%$$

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# CHAPTER V

## TRANSFORMERS IN SYSTEM STUDIES

In the usual system study, base voltage is arbitrarily chosen in a specified circuit; base voltages in the other circuits of the system are then determined by transformer turn ratios. There are systems, however, in which transformers of unequal turn ratios are operated in parallel or — a more usual condition — a transmission circuit is supplied at two or more different locations by transformers of unequal turn ratios, or of unequal equivalent turn ratios when several circuits are involved. For such cases, an equivalent circuit has been developed which is applicable to two-winding transformers when the ratio of transformation is different from the ratio of system base voltages on the two sides of the transformer.[1] This equivalent circuit is especially useful in the determination of load division when an a-c network analyzer is not available. Although developed for two-winding transformers, it has wider application and can be extended to transformers of more than two windings.

The usual procedure for representing such transformers on the a-c network analyzer is to use an autotransformer to take care of the difference in ratios between the turn ratio of the transformer and the ratio of the system base voltages. The autotransformer used has very low core loss and exciting current, and approximate compensation for its leakage reactance is secured by a series capacitor. For most problems it can be considered an ideal unit. The transformer leakage impedance may be placed on either side of the autotransformer; this impedance is expressed in per unit of system base ohms in the circuit in which it is placed.

### EQUIVALENT CIRCUITS FOR TRANSFORMERS WITH TURN RATIO DIFFERENT FROM THE RATIO OF SYSTEM BASE VOLTAGES AT THEIR TERMINALS

Equivalent circuits will be developed for two- and three-winding transformers with exciting currents neglected. It will be assumed that rated winding voltages are in direct proportion to the number of turns.

**Two-Winding Transformer.** With exciting current neglected, there will be no current in either winding with the other winding open. Let the two windings be designated 1 and 2. The proposed equivalent circuit is shown in Fig. 1 with branch impedances $Z_x$, $Z_y$, and $Z_z$ to be evaluated. In Fig. 1, the voltages at 1 and 2 referred to neutral $N$ represent the voltages across windings 1 and 2, respectively; the currents at 1 and 2 are the currents in windings 1 and 2, respectively, positive direction of current being from 1 to 2.

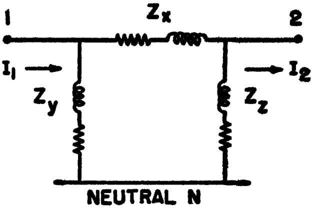

FIG. 1. Positive-sequence equivalent circuit (resonant Δ) for two-winding transformer bank, exciting current neglected, for use where transformer turn ratio differs from ratio of system base voltages at bank terminals. See [4]–[6] for evaluation of $Z_x$, $Z_y$, and $Z_z$.

Let $e_1$ be rated voltage of transformer winding 1 divided by base system voltage in winding 1. Let $e_2$ be a similar ratio in winding 2. The no-load voltage ratio of the transformer is then $e_1/e_2$ based on system base voltages.

Let $Z_t$ = transformer leakage impedance in per unit based on its rated winding voltages and system base kva per phase.

If the no-load voltage ratios in Fig. 1 are equated to those in the transformer based on system voltages,

$$\frac{Z_z}{Z_x + Z_z} = \frac{e_2}{e_1} \qquad [1]$$

$$\frac{Z_y}{Z_x + Z_y} = \frac{e_1}{e_2} \qquad [2]$$

With winding 1 short-circuited, the impedance viewed from winding 2 based on system base voltage in winding 2 is $e_2^2 Z_t$. If this value of impedance is equated to that in Fig. 1 viewed from 2 with 1 shorted to neutral,

$$\frac{Z_x Z_z}{Z_x + Z_z} = e_2^2 Z_t \qquad [3]$$

Simultaneous solution of [1]–[3] gives

$$Z_x = e_1 e_2 Z_t \qquad [4]$$

$$Z_y = \frac{e_1}{e_2 - e_1}(e_1 e_2 Z_t) \qquad [5]$$

$$Z_z = \frac{e_2}{e_1 - e_2}(e_1 e_2 Z_t) \qquad [6]$$

From [1] and [2], or from [4]–[6],

$$Z_x + Z_y + Z_z = 0 \qquad [7]$$

From [7], Fig. 1 is a resonant Δ since the sum of its branch impedances is zero. A resonant Δ could have been assumed in the first place and [7] used instead of [1] or [2] to evaluate $Z_x$, $Z_y$, and $Z_z$. With exciting current neglected, there will be no current flowing to neutral when a voltage is applied to one terminal with the other open. A resonant Δ satisfies this condition.

It will be noted from [4]–[6] that, if $e_1$ is greater than $e_2$, $Z_y$ will be negative; if $e_2$ is greater than $e_1$, $Z_z$ will be negative. If transformer resistance is included, the negative branch $Z_y$ or $Z_z$ will consist of a negative resistance and a capacitive reactance. Although the resonant Δ cannot be replaced by an equivalent Y of finite impedances, its branches can be combined with other system impedances in the usual manner to simplify calculations. In an analytic solution, negative resistances are as easily handled as positive.

If $e_1$ and $e_2$ are both unity, $Z_y = Z_z = \infty$ and Fig. 1 reduces to the series impedance $Z_x = Z_t$ = leakage impedance between windings.

If $e_1$ and $e_2$ are equal but not unity, as is the case when base voltages are in proportion to rated voltages but not equal to them, $Z_y$ and $Z_z$ become infinite, $e_1 = e_2 = e$, and the per unit leakage impedance $Z_x = e^2 Z_t$ is expressed in terms of base system quantities.

If base and rated voltage are the same in one of the windings, $e$ in that winding is unity. In many cases, base system voltage can be so chosen that $e_1$ or $e_2$ is unity.

**Problem 1.** Determine $Z_x$, $Z_y$, and $Z_z$ in per cent for insertion in the equivalent circuit of Fig. 1 for a transformer bank of three single-phase units each rated 9000 kva 121 kv/31.5 kv, with a reactance of 9% and a resistance of 0.9% based on its rating. System base three-phase kva is 30,000 kva; system base voltages in the circuits at the transformer bank terminals are 110 kv and 35 kv.

*Solution.* By definition,

$$e_1 = \frac{121}{110} = 1.10$$

$$e_2 = \frac{31.5}{35} = 0.90$$

$$Z_t = (0.9 + j9.0)\frac{10,000}{9000} = 1 + j10\%$$

In per cent, based on system base quantities,

$$Z_x = e_1 e_2 Z_t = 1.10 \times 0.90 Z_t = 0.99 + j9.90$$

$$Z_y = \frac{e_1}{e_2 - e_1} Z_x = \frac{1.10}{-0.20} Z_x = -5.5 Z_x = -5.445 - j54.45$$

$$Z_z = \frac{e_2}{e_1 - e_2} Z_x = \frac{0.90}{0.20} Z_x = 4.5 Z_x = 4.455 + j44.55$$

**Three-Winding Transformer with Base System Voltage in One Winding Different from Rated Voltage.** Let the three windings be designated 1, 2, and 3. With a three-winding transformer, it is reasonable to suppose that the arbitrarily chosen system base voltage will be such that in two of the windings (windings 1 and 2) system base voltages will be rated winding voltages. As rated voltages and base voltages are equal, employing the notation used for the two-winding transformer, $e_1 = e_2 = 1$. Base voltage in the third winding will be different from rated winding voltage, and $e_3$ will not be unity.

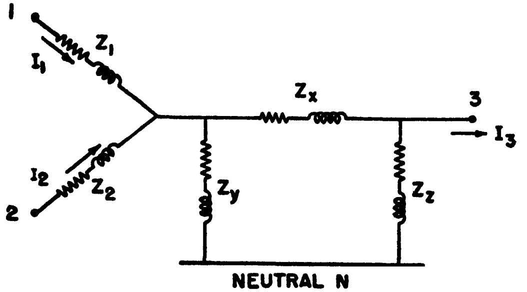

Fig. 2. Positive-sequence equivalent circuit for three-winding transformer bank, exciting current neglected, for use where transformer turn ratios differ from ratios of system base voltages at bank terminals.

Let $Z_{12}$, $Z_{13}$, and $Z_{23}$ be the per unit leakage impedances between windings indicated by the subscripts, taken two at a time with the other winding open, based on rated winding voltages and base system kva per phase.

To satisfy no-load conditions, there must be a resonant Δ in the equivalent circuit between terminals 1 and 3, and between 2 and 3, but not between 1 and 2. Figure 2 shows the proposed equivalent circuit, with branch impedances $Z_1$, $Z_2$, $Z_x$, $Z_y$, and $Z_z$ to be evaluated. It follows from the equivalent circuit of a two-winding transformer that

$$Z_y = \frac{1}{e_3 - 1} Z_x \qquad [8]$$

$$Z_z = \frac{e_3}{1 - e_3} Z_x \qquad [9]$$

The impedance between windings 1 and 2 with winding 3 open in the equivalent circuit of Fig. 2 must equal that in the transformer:

$$Z_1 + Z_2 = Z_{12} \qquad [10]$$

With terminal 3 shorted to neutral and voltage applied first at 1 with 2 open, and then at 2 with 1 open,

$$Z_1 + Z_x + \frac{Z_1 Z_x}{Z_y} = Z_{13} \qquad [11]$$

$$Z_2 + Z_x + \frac{Z_2 Z_x}{Z_y} = Z_{23} \qquad [12]$$

Simultaneous solution of [8]–[12] gives

$$\begin{aligned} Z_x &= \frac{Z_{13} + Z_{23} - Z_{12}e_3}{2} \\ Z_y &= \frac{Z_{12} + Z_{23} - e_3 Z_{12}}{2(e_3 - 1)} \\ Z_z &= \frac{e_3(Z_{12} + Z_{23} - e_3 Z_{12})}{2(1 - e_3)} \\ Z_1 &= \frac{Z_{12}e_3 + Z_{13} - Z_{23}}{2e_3} \\ Z_2 &= \frac{Z_{12}e_3 + Z_{23} - Z_{13}}{2e_3} \end{aligned} \qquad [13]$$

If $e_3 = 1$ in [13], $Z_y$ and $Z_z$ become infinite, and the per unit equivalent circuit is reduced to that of the three-winding transformer based on rated winding voltages and system kva. See [5], Chapter IV.

The equivalent circuits for two- and three-winding transformers given in Figs. 1 and 2 can be used to replace a transformer bank of identical single-phase units or a three-phase transformer in the positive- and negative-sequence networks when ratios of system base voltages in the windings are not the same as the corresponding turn ratios.

Zero-sequence equivalent circuits depend upon the manner in which the windings are connected and the method of grounding; however, they can be derived from Figs. 1 and 2 by the modifications given and illustrated in Chapter IV for banks of single-phase units. In a transformer bank of single-phase units, the impedance to positive-, negative-, and zero-sequence currents is the same, provided that there is a path for zero-sequence current. In a three-phase transformer, the zero-sequence leakage impedances between windings may differ appreciably from the positive. (See " Three-Phase Transformers," Chapter IV.)

## OPEN CONDUCTORS IN CIRCUITS SUPPLYING UNGROUNDED TRANSFORMERS

In the equivalent circuits of Figs. 1 and 2, and in many of those in Chapter IV, exciting currents are neglected. This can be done when the exciting impedance is so large relative to the other impedances under consideration that it may be regarded as infinite within the degree of precision required in calculations. Such is usually the case in short-circuit calculations where there are no open phases and short-circuit currents are large relative to transformer exciting currents.

When one or two conductors are open in circuits which supply ungrounded transformers, the conditions may be quite different. The transmission circuit may be a low-voltage feeder of short length and the transformer bank of low kva rating, so that the capacitive reactances of the circuit and the magnetizing reactances of the transformer bank are infinite relative to the resistances and other reactances of the system. But the capacitive reactances are negative, and the magnetizing reactances are positive; therefore, for certain ratios of these reactances a current path of relatively low impedance may be provided with resultant high voltages on the open phase or phases. If such is the case, the resistances and other reactances of the system may be so small relative to the capacitive reactances of the circuit and the magnetizing reactances of the transformer bank that they can be regarded as zero within the degree of precision required in calculations.

Under certain conditions, high sustained voltages have resulted from the opening of one or two conductors in circuits which supply ungrounded transformer banks. Voltages of sufficient magnitude to damage equipment have been reported, and induction motors have been observed to reverse their direction of rotation. Cases of the breakage of a line conductor with resulting high sustained voltages have been noted, and abnormal voltage conditions have occurred on potential transformers.

Open conductors may be due to the blowing of fuses, operation of single-pole switches, or of any interrupting device in which the interval of time between the opening or closing of the first and last phases is of sufficient length to permit ultimate steady-state voltage conditions to be attained. Included also is the accidental breaking of a conductor where one severed end may fall to ground.

To determine the conditions under which abnormal voltages may occur and motors reverse their direction of rotation, with one or two conductors open in circuits supplying ungrounded transformers, an investigation was made by means of calculations and laboratory tests, and the results were reported.[2]

The phenomena encountered may be summarized as follows: When one conductor of a three-phase circuit supplying an ungrounded, unloaded transformer bank is open, there is a path for currents from the closed conductors through the exciting impedance of the bank and the capacitance between conductors to the open conductor, and thence to ground through the capacitance-to-ground of the open conductor; see Fig. 5(*a*). There is a similar path with two conductors open; see Fig. 5(*b*). When the three-phase system is grounded, the path is completed through the system grounded neutrals; if ungrounded, through the capacitance-to-ground on the system side of the opening.

With the inductive reactance of the transformer bank paralleled by capacitance between conductors in series with line capacitance to ground, high voltage to ground at the transformer bank terminal or terminals of the open phase or phases will result for certain ratios of line capacitive reactances to transformer magnetizing reactances. The capacitances of the transformer bank, bushings, and bus structure, with no intervening length of line, may be sufficient to produce high voltages on small banks, such as potential transformers. On the other hand, there are many circuits with so little capacitance, or such high capacitive reactances relative to transformer magnetizing reactances, that no voltages above normal will occur when one or two conductors are open. As a matter of interest, a fused overhead circuit of this nature, supplying ungrounded transformers, was frequently out of service because of limbs of trees falling upon it; otherwise, operation was satisfactory. To eliminate this cause of outages, the overhead circuit was replaced by an underground cable, thereby materially increasing the capacitance of the circuit or decreasing its capacitive reactance. The result was unfortunate; the first time a fuse blew, the circuit was subjected to sustained overvoltages, dangerous to equipment.

In general, the lower the ratio of the capacitive reactances of a circuit between opening and transformers to normal transformer magnetizing reactance, the more likely are overvoltages to occur. The question to be answered is, how high must this ratio be to avoid dangerous voltages in circuits supplying ungrounded transformers through fuses on single-pole switches?

The following cases will be considered.

Case I. Three-phase power source solidly grounded.

A. One conductor open. No Fault.
B. Two conductors open. No Fault.

C. One or two conductors open and faults to ground on either side of the open conductor or conductors.

Case II. Three-phase power source ungrounded.

A. One conductor open. No Fault.

B. One conductor open and fault to ground on system side of open conductor.

C. One conductor open and fault to ground on transformer side of open conductor.

In all cases the primary windings of the transformer bank are ungrounded.

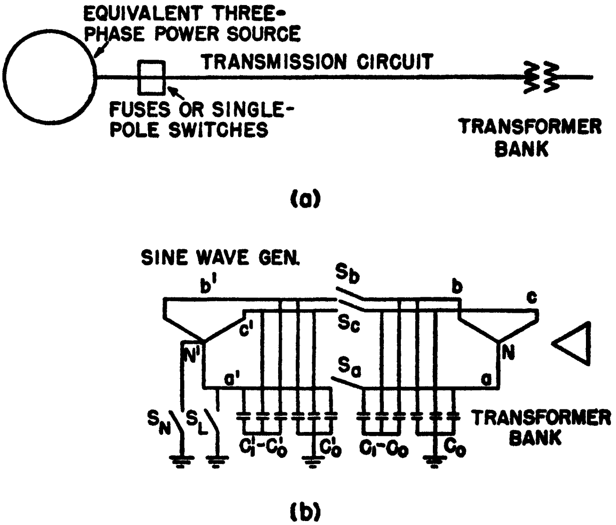

Fig. 3. System studied. (*a*) One-line diagram. (*b*) Miniature system equivalent circuit with one or two open conductors and grounded or ungrounded power source.

**Simplified System.** A one-line diagram of the system studied is given in Fig. 3(*a*), which consists of a three-phase power source supplying a transmission circuit and transformer bank through fuses or single-pole switches. It will be assumed that the three-phase system is large relative to the kva rating of the bank and therefore can be represented by an equivalent generator with balanced line-to-line voltages and negligible positive- and negative-sequence impedances. If the system is grounded, the equivalent generator will also have balanced line-to-ground voltages, and its zero-sequence impedance will be negligible. The three-line diagram of Fig. 3(*a*) is given in Fig. 3(*b*). As the highest voltages are apt to occur when the transformers are unloaded or lightly loaded, the study was made with the bank unloaded. The sine-wave generator on the miniature system[3] used in laboratory tests has a very low impedance relative to the exciting reactance of the transformer bank and the line capacitive reactances used so that balanced line-to-line voltages are applied to the circuit at $a'b'c'$ under all conditions. The system becomes a grounded one with balanced line-to-ground voltages when switch $S_N$ is closed. Resistance and reactance are neglected in the transmission circuit since they are insignificant relative to the capacitive reactances for the lengths of line under discussion. $C_1'$ and $C_0'$ represent the positive- and zero-sequence capacitances, respectively, on the system side of the fuses or single-pole switches; $C_1$ and $C_0$ represent the positive- and zero-sequence capacitances, respectively, of the circuit supplying the ungrounded, unloaded transformer bank which is shown in Fig. 3(*b*) as Y-connected on the circuit side and Δ-connected on the load side.

With the system set up on a three-phase basis as in Fig. 3(*b*), $C_1$ and $C_1'$ represent both positive- and negative-sequence capacitances, and the ground is a point of zero potential in the positive- and negative-sequence systems. The zero-sequence capacitances $C_0$ and $C_0'$ are therefore positive- and negative-sequence capacitances as well. (See Volume I, page 452.) The remainders of the positive- and negative-sequence capacitances $C_1 - C_0$ and $C_1' - C_0'$ are placed in ungrounded Y's in Fig. 3(*b*) which offer infinite impedances to zero-sequence currents.

**Transformers.** In the miniature system, voltage conditions were obtained with transformer banks made up of three single-phase units and with three-phase transformers of both the shell type and the core type, the windings on the line side and also the secondaries being connected alternately in Δ and Y for each case. The miniature system single-phase transformers were operated at a normal voltage corresponding to the saturation curve shown by curve 1 of Fig. 4. Curve 2 gives a transformer saturation curve with a slightly higher degree of saturation than curve 1 of the miniature system Curve 5, for the specially built miniature system transformers, gives an intermediate value of saturation and because of its higher impedance, makes it possible to cover a greater range of system constants. The saturation curves of the three-phase core-type and shell-type transformers of the miniature system are given by curves 3 and 4, respectively, of Fig. 4. These units are reproductions to scale of typical power transformers. All saturation curves give applied sinusoidal rms voltage versus

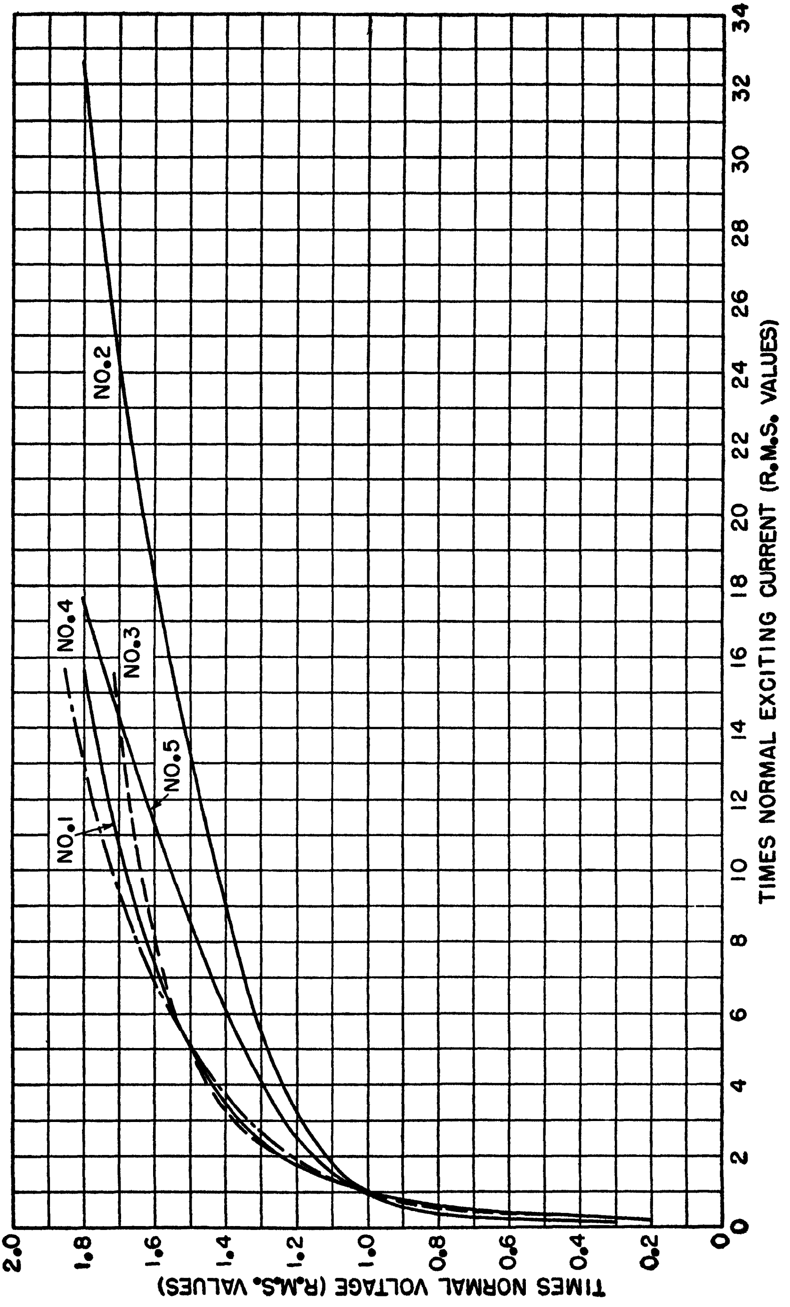

Fig. 4. Transformer saturation curves used in making calculations and in the miniature system.

exciting rms current, with voltage and current expressed in terms of normal voltage and normal exciting current, respectively. They are similar to the excitation curves of Fig. 2, Chapter IV, already described.

The effects of transformer saturation were taken into account in calculations by using transformer banks having different degrees of saturation and different excitation curves. Excitation curves 1 and 2 of Fig. 4 were used for the single-phase transformers of the bank.

Let $x_m$ = per unit transformer magnetizing reactance per phase, based on transformer rated kva and normal voltage. At any voltage different from normal, the magnetizing reactance can be determined from the saturation curve (see Problem 1, Chapter IV). In any given case it is necessary to know $x_m$ and the saturation curve.

## Case I. Power Source Solidly Grounded

Under the assumption of a symmetrical system replaced by an equivalent synchronous generator with grounded neutral and balanced generated voltages, the charging currents taken by $C_1' - C_0'$ and $C_0'$ will be normal charging currents. Under the additional assumption of negligible reactances and resistances in the equivalent generator and distribution circuit, any currents flowing in $C_1' - C_0'$ and $C_0'$ or from phases $b$ and $c$ through $C_0$ to ground will have no effect upon the voltages $V_b$ and $V_c$ of the closed phases, or upon the voltage of phase $a$ on either side of the opening. Under the stated assumptions, the voltages $V_b$ and $V_c$ at the transformer terminals in phases $b$ and $c$ are equal to the generated voltages $E_b$ and $E_c$, respectively, and the voltage $V_a'$ of phase $a$ on the system side of the opening is equal to the generated voltage $E_a$. The three-line diagram of Fig. 3($b$) reduces to that of Fig. 5($a$) for one open conductor and to Fig. 5($b$) for two open conductors. In Fig. 5($a$), the transformer bank is shown Δ-connected on the line side and Y-connected on the load side. In Fig. 5($b$), it is Y-connected on the line side and Δ-connected on the load side. All possible connections were considered.

**Capacitances.** In overhead circuits the ratio of zero-sequence capacitance $C_0$ to positive-sequence capacitance $C_1$ will be greater than 0.5 and less than unity. In cable circuits it will be greater than 0.5 and may be as high as unity. (See Volume I, Chapter IX, and Volume II, Chapter III.) To include all ratios and show as great a variation as practical for $C_0/C_1$, tests and calculations were made with $C_1 = C_0$ and $C_1 = 2C_0$.

For a known ratio of $C_0$ to $C_1$, the voltage to ground of the open conductor or conductors on the transformer side of the opening in a grounded system will be expressed in terms of $x_{c1}/x_m$ where

$x_m$ = per unit magnetizing reactance per phase at normal voltage of the transformer bank

$x_{c1}$ = per unit positive-sequence capacitive reactance to neutral of the circuit between opening and transformer bank

$x_m$ and $x_{c1}$ are based on the same kva per phase.

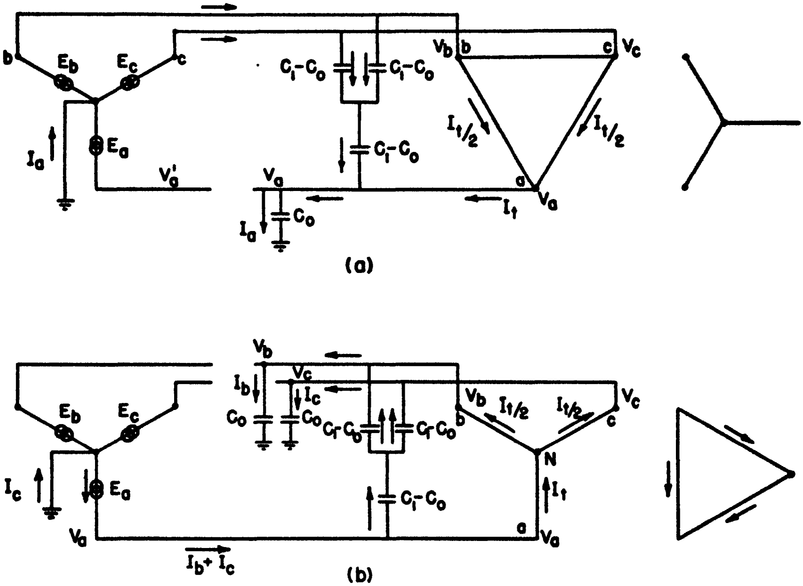

FIG. 5. Solidly grounded system. (*a*) One conductor open. (*b*) Two conductors open. Arrows indicate path of current.

Although only one transformer bank or three-phase transformer was used in calculations and tests, the results can be applied to a group of transformers. In more general terms, $x_m$ is the equivalent normal magnetizing reactance of the total transformer kva on the load side of the fuse or single-pole switch location; likewise $x_{c1}$ and $x_{c0}$ correspond to the total positive- and zero-sequence capacitances $C_1$ and $C_0$, respectively, on the load side of the open conductor or conductors.

From Figs. 5(*a*) and 5(*b*) it can be seen that for a transformer bank of three single-phase units, the fundamental-frequency voltages to ground on the unfaulted phases with transformer resistances neglected depend only upon $x_m$, $C_1$, $C_0/C_1$, the saturation curve of the transformers, and the manner of connecting the windings. [It is suggested that the student at this point develop equations for the fundamental-frequency voltages to ground on the open phases in Figs. 5(*a*) and 5(*b*) in terms of $x_m/x_{c1}$ for an assumed ratio of $C_0/C_1$, neglecting transformer resistances, and (*a*) assuming no saturation, (*b*) using the saturation curve of Fig. 2, Chapter IV.]

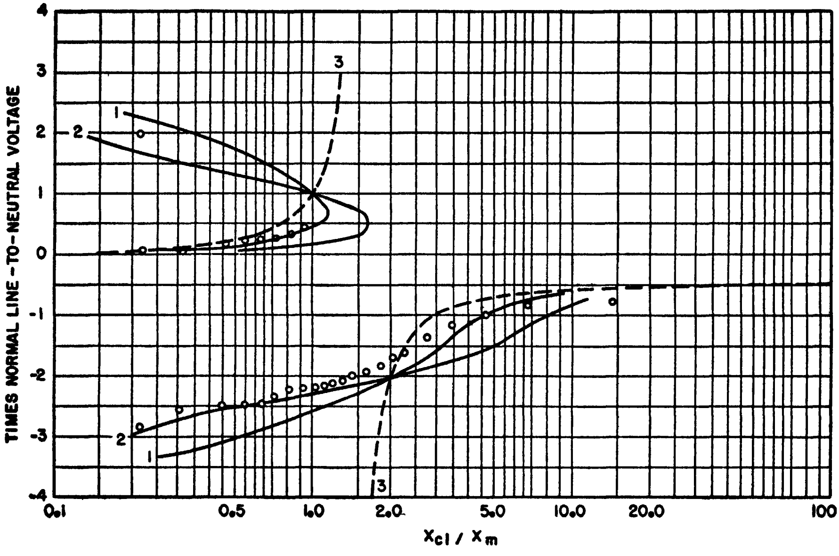

FIG. 6(*a*). Steady-state maximum voltages on open phase with one conductor open. No fault on system. Power source solidly grounded. $C_1 = C_0$. Transformers Δ-connected on line side and Δ- or Y-connected on load side. Curve 1 — saturation curve 1, Fig. 4; curve 2 — saturation curve 2, Fig. 4; curve 3 — no saturation; O — test points, miniature system with saturation curve 1, Fig. 4. *Note:* With Δ-connected primary windings, $x_m$ is in per unit of base Δ ohms per phase while $x_{c1}$ is in per unit of base *Y* ohms per phase. If the ratio $x_{c1}/x_m$ is calculated with $x_m$ and $x_{c1}$ in ohms, this ratio should be divided by 3 for Δ-connected primary windings before entering the charts.

In laboratory tests on a miniature system, harmonics and transformer resistances are automatically included, and the effect of various transformer connections are readily taken into account.

Curves giving results of calculations and tests on the miniature system will be discussed before the method used in making calculations is explained.

**A. One Conductor Open. No Fault.** The curves of Fig. 6(*a*) give the steady-state maximum peak voltage to ground of the open conductor on the transformer side of the opening in terms of normal line-to-neutral peak voltage with the transformer bank Δ-connected on the line side, and Y- or Δ-connected on the load side, versus $x_{c1}/x_m$ for $C_0 = C_1$. Curves 1, 2, and 3 were calculated by consideration only of fundamental-frequency components of current and voltage (the method to be explained later). For curves 1 and 2 transformer saturation curves 1 and 2, respectively, of Fig. 4 were used. Curve 3 is

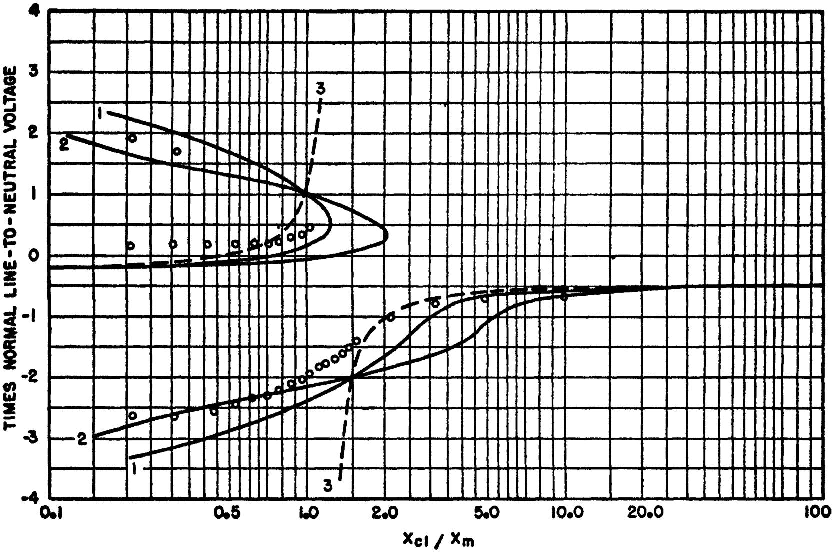

FIG. 6(b). Steady-state maximum voltages on open phase with one conductor open. No fault on system. Power source solidly grounded. $C_1 = 2C_0$. Transformers Δ-connected on line side and Δ- or Y-connected on load side. [See note to Fig. 6(a)]. Curve 1 — saturation curve 1, Fig. 4; curve 2 — saturation curve 2, Fig. 4; curve 3 — no saturation; O — test points, miniature system with saturation curve 1, Fig. 4.

for no saturation. Points obtained by test on the miniature system, with transformer saturation curve 1 of Fig. 4, are indicated by circles.

The reference vector for calculated voltages is the voltage of the open phase on the system side of the opening, so that with no capacitance in the circuit, or $x_{c1} = x_{c0} = \infty$, the voltage on the transformer side of the opening is $-0.5$, and the voltage at the load is single-phase. As the ratio $x_{c1}/x_m$ is decreased, the voltage of the open phase increases in magnitude in the negative direction and there is phase reversal at the load. At $x_{c1}/x_m = 2$ for all calculated curves, and about 1.5 for the test curve, the voltage-to-ground of the open phase is $-2$, and the

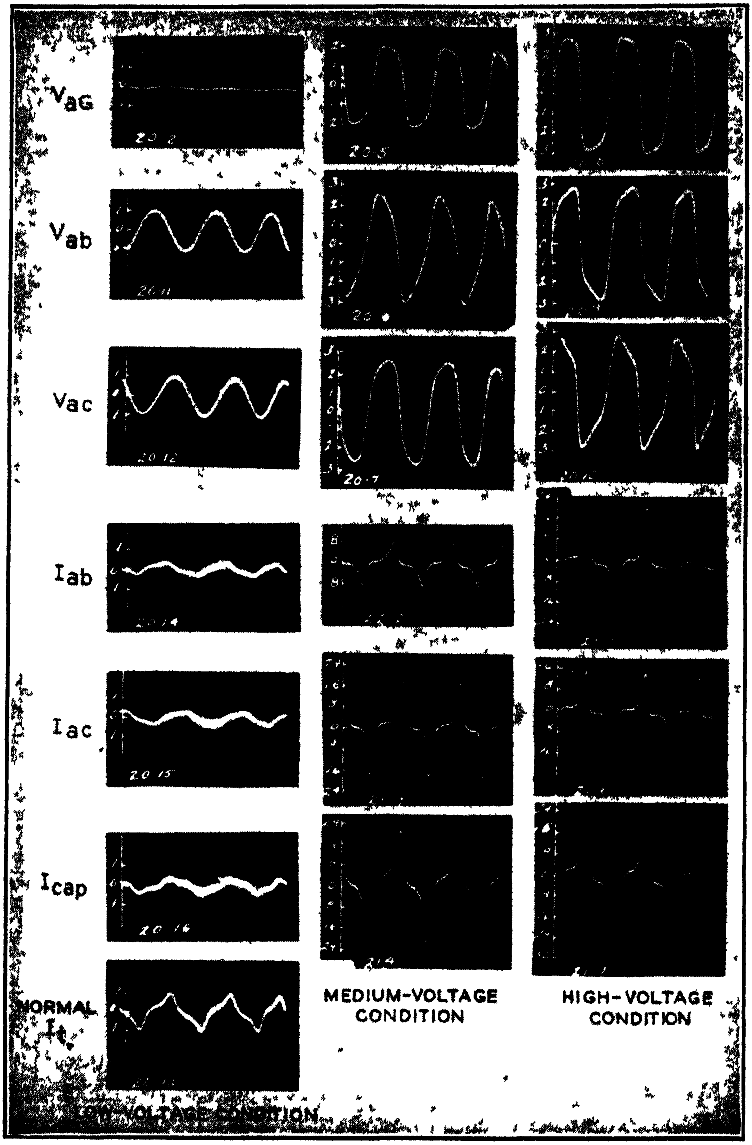

FIG. 7. Δ-connected load transformer; $C_0 = C_1$; $x_{cl}/x_m = 0.212$. Calibration: current — times normal crest transformer exciting current; voltage — times normal line-to-neutral crest voltage.

line-to-line voltages at the load are balanced normal voltages of reverse phase rotation. (For the calculated curves, $V_a = -2$, $V_b = -\frac{1}{2} - j\sqrt{3}/2$, $V_c = -\frac{1}{2} + j\sqrt{3}/2$.) As $x_{c1}/x_m$ is further decreased the voltage to ground of the open phase continues to increase and remains negative on both calculated and test curves; but before $x_{c1}/x_m = 1$ is reached, two other steady-state values of voltage are possible as indicated by the calculated curve. Both are positive. Only one of these other indicated values, the lower of the two, was obtained in test, until $x_{c1}/x_m$ was decreased to about 0.2, but for values of $x_{c1}/x_m = 0.2$ or lower, three different values were obtained by test. Figure 7 shows oscillograms of the voltages and currents for the three possible stable conditions, any one of which may occur, depending upon the phases of the voltages at the instant the switch is opened. With the switch opened at random a number of times, the high negative voltage or the low positive occurred more often than the medium positive voltage.

Figure 6(*b*) is similar to Fig. 6(*a*) except that $C_1 = 2C_0$. The voltages above normal of Fig. 6(*b*) are lower than those of Fig. 6(*a*) for the same values of $x_{c1}/x_m$.

Curves for single-phase transformers, Y-connected on the line side and Δ- or Y-connected on the load side, and for core-type or shell-type three-phase transformers do not differ materially from the curves of Figs. 6(*a*) and 6(*b*), provided the transformer saturation curves are the same.

**B. Two Conductors Open. No Fault.** The curves of Fig. 8 give the calculated steady-state maximum peak voltage to ground of the two open conductors on the transformer side of the opening in terms of normal line-to-neutral peak voltage versus $x_{c1}/x_m$. Curve 1 is for $C_1 = C_0$; curve 2 is for $C_1 = 2C_0$. Saturation curve 1 of Fig. 4 was used for both curves, with transformers Δ-connected on the line side and Y-connected on the load side. Test points were obtained from the miniature system with single-phase transformers having saturation curve 1 of Fig. 4, for $C_1 = C_0$ and $C_1 = 2C_0$, and transformer connections given under Fig. 8.

With the voltage of the closed phase *a* as reference vector and no capacitance in the circuit, or $x_{c1} = x_{c0} = \infty$, the voltage of the two open conductors *b* and *c* on the transformer side of the opening are unity and positive. As $x_{c1}/x_m$ is decreased, these voltages increase in magnitude and remain positive; but for values of $x_{c1}/x_m$ slightly above unity and below unity, calculations indicate that two other values of voltage are possible, both of less magnitude than the first discussed. The shape and extent of these lower-voltage curves depend largely upon the transformer saturation curve.

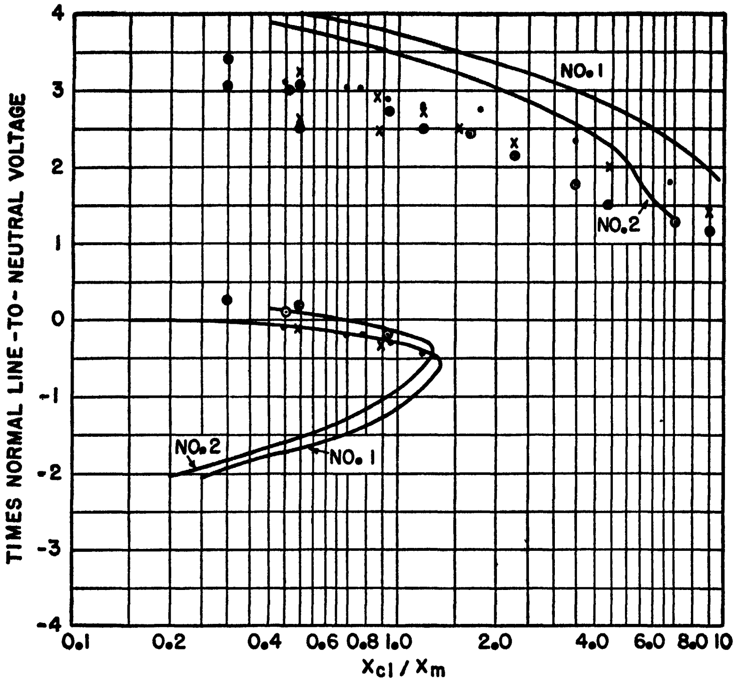

FIG. 8. Steady-state voltages on open phase with two conductors open. No fault on system. Power source solidly grounded. Curve 1 — calculated using curve 1, Fig. 4, for $C_1 = C_0$ and transformers Δ-connected on line side and Y-connected on load side; curve 2 — same but $C_1 = 2C_0$. [See note to Fig. 6(*a*)]. Test points, miniature system using saturation curve 1, Fig. 4. Single-phase transformers. ● — $C_1 = C_0$, transformers Δ line Y load; ⊙ — $C_1 = 2C_0$, transformers Δ line Y load; x — $C_1 = C_0$, transformers Y line Y or Δ load; ⊗ — $C_1 = 2C_0$, transformers Y line Y or Δ load.

Maximum voltages are considerably higher with two conductors open than with one; and voltages above normal occur at higher ratios of $x_{c1}/x_m$.

**C. Open Conductors and Ground Faults.** With one open conductor and a line-to-ground fault on the system side of the opening, the voltage to ground of the open conductor on the transformer side of the opening is in no way affected by the presence of the fault. In this case, Figs. 6(*a*) and 6(*b*) apply. If the line-to-ground fault occurs on the transformer side of the open conductor, the voltage to ground there is reduced to zero, and no resultant overvoltages will occur. With two open conductors and a double line-to-ground fault on the system side of the opening, Fig. 8 applies. With the fault on the transformer side of the opening, there will be no resultant overvoltages.

## Case II. Ungrounded Power Source

**A. One Conductor Open. No Fault.** If the zero-sequence capacitance $C_0'$ on the system side of the opening is large relative to the capacitances $C_0$ and $C_1$ of the circuit on the transformer side of the opening, the system is effectively grounded. The curves in Figs. 6(*a*) and 6(*b*) for the grounded system then apply. For values of $C_0'$ equal to or less than $C_1$, voltages to ground of the open phase on the transformer side of the opening are less than those of a solidly grounded system.

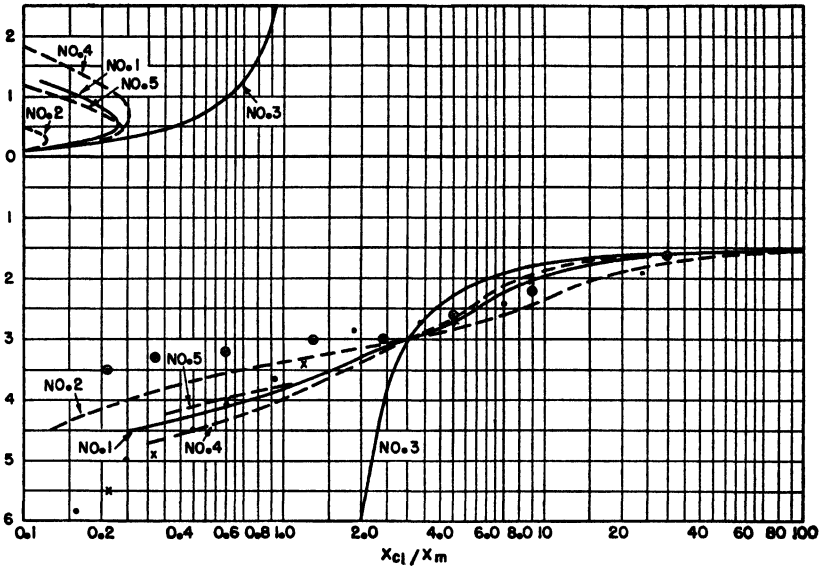

FIG. 9. Maximum steady-state voltages to ground with one conductor open. Line-to-ground fault on source side of open conductors. System neutral isolated. $C_1 = C_0$. These results are independent of $C_1'$ and $C_0'$. No load on system.

Single-phase transformers, Δ-connected on line side, Δ- or Y- on load side [see note to Fig. 6(*a*)]: curve 1, calculated voltages based on saturation curve 1, Fig. 4; curve 2, calculated voltages based on saturation curve 2, Fig. 4; curve 3, calculated voltages based on no saturation; ●, test points, miniature system saturation curve 1, Fig. 4.

Single-phase transformers, Y-connected on line side: curve 4, calculated voltages based on saturation curve 1 of Fig. 4 with Y-connected load side; curve 5, same as 4 but Δ-connected load side; ⊙, test points, miniature system, saturation curve 1, Fig. 4, Δ-connected load side; ·, same but Y-connected load side.

**B. One Conductor Open and Fault to Ground on the System Side of the Opening.** The range of voltages for various transformer connections are shown in Fig. 9. Curves 1, 2, and 3 are for banks of single-phase transformers Δ-connected on the line side and Δ- or Y-connected on the load side. For curve 4, they are Y-connected on both sides; for curve 5, Y-connected on the line side and Δ-connected on the load side. Saturation curve 2 of Fig. 4 was used for curve 2. Curve 3 is for no saturation. For curves 1, 4, and 5, saturation curve 1 of Fig. 4 was used. Test points from the miniature system were obtained with saturation curve 1 of Fig. 4 with the transformer connections indicated under Fig. 9. From the test points, it can be concluded that transformer connections make little difference for ratios of $x_{c1}/x_m$ greater than unity. For very low values of $x_{c1}/x_m$, the Y-Δ connection reduces the overvoltages in this region.

Figure 9 gives the highest voltages that were obtained. It is of interest in this connection to note that the V.D.E. Standard,[4] in effect since October 1, 1925, points out the danger of using fuses and single-pole disconnecting switches under certain conditions, and states that the breakage of a conductor, where the end on the system side falls to ground, may give sustained voltages to ground on the portion of the conductor severed from the main system as high as $3\sqrt{3}$ times normal line-to-neutral voltage.

**C. One Conductor Open with Fault to Ground on the Transformer Side of the Opening.** Under this condition, Fig. 10 gives the voltages to ground of the open conductor on the system side versus the ratio $x_{(c_0+c_0')}/x_m$ with $x'_{c0}/x_{c0}$ as parameter, where $x_{(c_0+c_0')}$ is the total zero-sequence capacitive reactance on both sides of the opening. The calculated curves in Fig. 10 were obtained with single-phase transformers Δ-connected on the line side and Y- or Δ-connected on the load side, and excitation curve 1 of Fig. 4. Test points are for saturation curves 1 and 5 of Fig. 4 and values of $x_{c_0'}/x_{c0}$ indicated under Fig. 10.

If the ratio of total zero-sequence capacitive reactance (on both sides of the opening) to $x_m$ is large, no high voltages are produced. As this ratio is decreased, voltages are increased. Only one voltage value is obtained until the ratio of total zero-sequence capacitive reactance to magnetizing reactance, $x_{(c_0+c_0')}/x_m$, reaches a value of about 5.0. Below this value, three stable voltage conditions were obtained by test in some cases. As the capacitive reactance was decreased, increasingly higher maximum voltages were obtained. With a relatively low-loss transformer (curve 1 of Fig. 5), the high-voltage condition could be obtained for values of $x_{(c_0+c_0')}/x_m$ down to 0.03, although the low-voltage condition occurred more often when the switch was opened at random a number of times.

With the smaller transformers (curve 5 of Fig. 5) which had slightly higher losses, the high-voltage condition did not occur for values of $x_{(c_0+c_0')}/x_m$ below 0.2, approximately. This elimination refers only to the steady-state condition, however. In many cases, long transient conditions were observed which could be quite hazardous to insulation or connected apparatus.

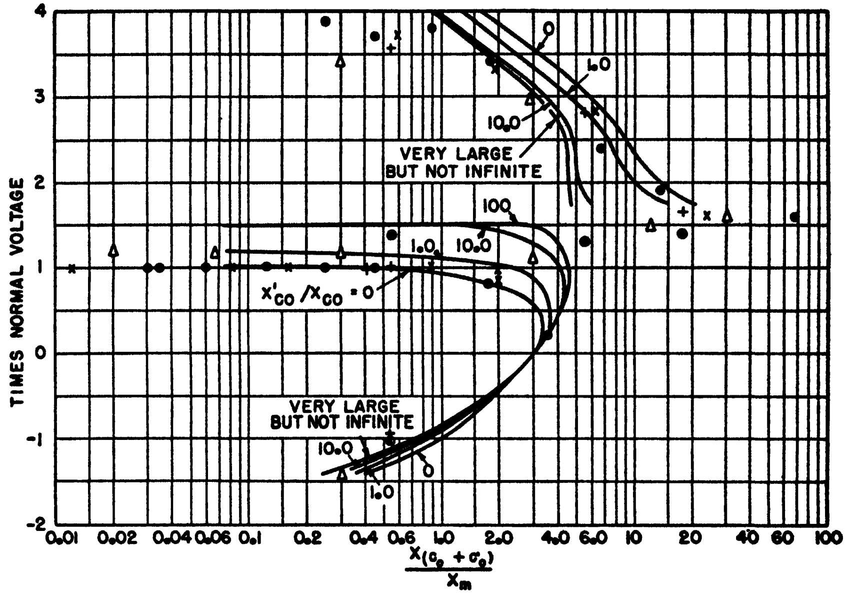

Fig. 10. Isolated neutral system. Steady-state voltages on open phase with one conductor open. Fault to ground on load side of open conductor. Saturation curve 1 of Fig. 4 used for calculated curves. ⊙, test points for $x'_{c0}/x_{c0} = 0$, and saturation curve 1, Fig. 4. Test points for $x'_{c0}/x_{c0} = 0, 0.1, 1.0$, and 10.0, and saturation curve 5, Fig. 4, indicated by x, +, Δ, and ⊗, respectively.

**Discussion and Summary.** The manner of connecting the windings of an ungrounded transformer bank, whether Δ-Δ, Δ-Y, Y-Y, or Y-Δ, does not materially affect the magnitudes of the voltages to ground obtained on the open phases except in the very high-voltage region. The voltages to ground of the open phases obtained with three-phase transformers of both core type and shell type are approximately the same as those obtained with banks of single-phase units, with the same saturation curve, for corresponding values of $x_{c1}/x_m$.

Calculations based on consideration of only fundamental-frequency components of voltage and current give sustained voltages to ground which approximately check tests made on the miniature system.

Except where the calculated voltages are less than normal, the method gives values which, with a few exceptions in the very high-voltage region, are too high. The method can therefore be considered to give conservative values.

While test points agree quite well with calculated results, it may be well to point out that, for some of the conditions which gave rise to very high voltages, it was possible to obtain slight variations in the wave shape depending upon the initiating conditions. For instance, the high voltage might, for one opening of the switch, be essentially of fundamental frequency. For a second opening of the switch, it might be essentially of the same magnitude, but contain definite subharmonics. Some of these subharmonics observed were of very low frequency.

*For the grounded system* or the isolated system effectively grounded through its capacitance to ground, a fault to ground on the open conductor, or conductors, on the system side of the opening, or openings, does not affect the sustained voltages at the transformer bank terminals. A fault to ground on the transformer side of the opening reduces the voltage there to zero. The highest voltages are obtained when two conductors are open and there is no fault. Such a condition can exist when a line-to-line fault is cleared by the blowing of two fuses; and also on a circuit controlled by single-pole switches when one phase is closed before the other two.

If the peak value of the sustained voltage to ground of the open conductor on the load side of the opening in a solidly grounded system with ungrounded transformer bank is to be kept below 1.73 times normal line-to-neutral peak voltage with two conductors open, the ratio $x_{c1}/x_m$ (from Fig. 8) must not be less than approximately 6. From Fig. 8 it may be seen that calculated voltages in the region of 1.73 times normal line-to-neutral voltage are greater than test voltages. Allowing for this, and with $C_1 > C_0 > (C_1/2)$, voltages above 1.73 times normal will probably not occur with transformers of the usual degree of saturation if the ratio $x_{c1}/x_m$ is 6 or greater. With one conductor open, from Figs. 6(*a*) and 6(*b*), $x_{c1}/x_m$ should be 3 or greater.

With a criterion $x_{c1}/x_m = 6$, the length of line $\ell$, between the opening and the transformer bank, should not exceed a value which can be approximately determined from the normal voltage of the line and the rated kva and exciting current of the transformers.

The positive-sequence capacity susceptance of overhead transmission circuits, $2\pi f C_1$, in micromhos per mile at 60 cycles, varies from about 5 to 7, depending upon the diameter of the conductors and the spacing between them. Since these, in turn, depend upon voltage,

$2\pi fC_1$ at a given voltage and frequency can be estimated with fair accuracy. The following values will be taken as typical at 60 cycles:

| Kv | $2\pi fC_1$ Micromhos per Mile |
|---|---|
| 230–115 | 5.2 |
| 69–34.5 | 5.5 |
| 34.5–13.8 | 6 |
| Below 13.8 | 6.5 |

The zero-sequence capacitance of overhead transmission circuits is less than the positive, but usually greater than one-half the positive.

With $x_{c1}/x_m = 6$ as the criterion,

$$x_{c1}/x_m = I_m x_{c1} = I_m \frac{10^6}{2\pi fC_1 \ell} \times \frac{\text{kva}}{\text{kv}^2\, 10^3} = 6$$

$$\ell = I_m \frac{\dfrac{\text{kva } 10^3}{\text{kv}^2}}{6(2\pi fC_1)}$$

where $\ell$ = allowable length of line between fuses or single-pole switches and load transformer bank

kva = rated kva of the transformer bank

kv = normal line-to-line voltage of the circuit in kilovolts

$I_m$ = average per unit rms exciting current at normal voltage

$2\pi fC_1$ = positive sequence capacitive susceptance of the circuit in micromhos per mile.

Transformer exciting current $I_m$ lies between 1% and 10%, with 3.5% not an unusual value. Figure 11, drawn with $I_m = 3.5\%$, gives the approximate maximum allowable length of line at various voltages supplying a transformer bank of given kva rating and 3.5% average rms magnetizing current which can be operated by fuses or single-pole switches without the risk of voltages to ground exceeding line-to-line voltages when two conductors are open. This allowable length of line varies directly as the per unit rms exciting current of the bank and inversely as the ratio $x_{c1}/x_m$ selected as criterion. For an rms exciting current of 7%, the allowable lengths of line would be doubled; for an exciting current of 1%, they would be only about one-third those of Fig. 11.

Since, for some cases, the maximum allowable length of line is very short, it may be necessary under such conditions to take into account the internal capacitance of the transformer bank itself. Generally this will not have to be done. For cases requiring this refinement, representative values of internal capacitance of power transformers are given by L. V. Bewley.[5] The largest values of capacitance given are on the order of 0.01 microfarad, which would be equivalent to one mile of line, or less under most conditions. For the higher voltages, the

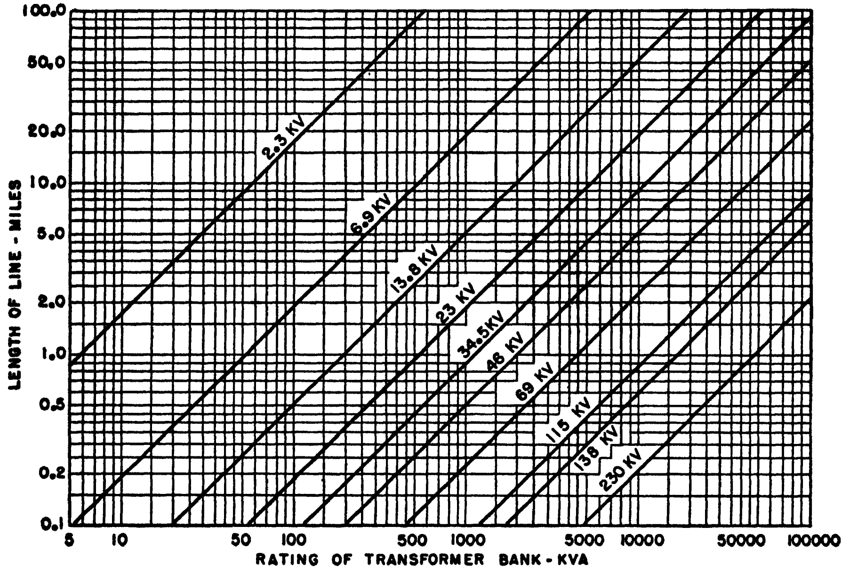

FIG. 11. Solidly grounded system, approximate maximum allowable length of line between fuses or single-pole switches and load transformer to limit line-to-ground voltage on open phase to $\sqrt{3}$ times normal, with two conductors open: $x_{c1}/x_m = 6$; $I_m = 3.5\%$ rms average exciting current. *Note:* Allowable length of line varies directly with rms average exciting current.

internal capacitance would be equivalent to considerably less than a mile of overhead line. Bushing capacitance, being very small relatively, can be neglected.

In case the transmission circuit contains some cable, its equivalent in terms of miles of overhead line may be obtained approximately from Fig. 12. (This figure of reference 2 was reproduced from reference 6.)

*For the ungrounded system,* the highest voltages appear with the simultaneous occurrence of an open conductor and line-to-ground fault, such as might correspond to the breaking and falling to ground of a line conductor. The criterion for the solidly grounded system is adequately conservative for conditions not involving faults. However, should a permanent fault occur on the system side, the criterion must be changed if voltages are to be limited to maximum values of $\sqrt{3}$ times normal. Figure 9 shows that a minimum ratio of $x_{c1}/x_m = 25.0$ approximately is required. This criterion alone is not sufficient, however. If a permanent fault on the load side is assumed, the total zero-sequence capacitance of the system must be very large or else very small in order to prevent overvoltages. See Fig. 10. Therefore, in addition to imposing the criterion of $x_{c1}/x_m \geqq 25.0$, restrictions must also be imposed on the ratio of $x'_{c0}/x_m$. From Fig. 10, the ratio of $x_{(c_0+c'_0)}/x_m$ should be less than approximately 0.04 or greater than a value varying from 4.7 to 25.0 approximately, depending upon the ratio $x'_{c0}/x_{c0}$. In the light of our first restriction on $x_{c1}/x_m$, this can be conservatively interpreted to impose the restriction that $x'_{c0}/x_m$ must be less than 0.04 or greater than 40.0, approximately.

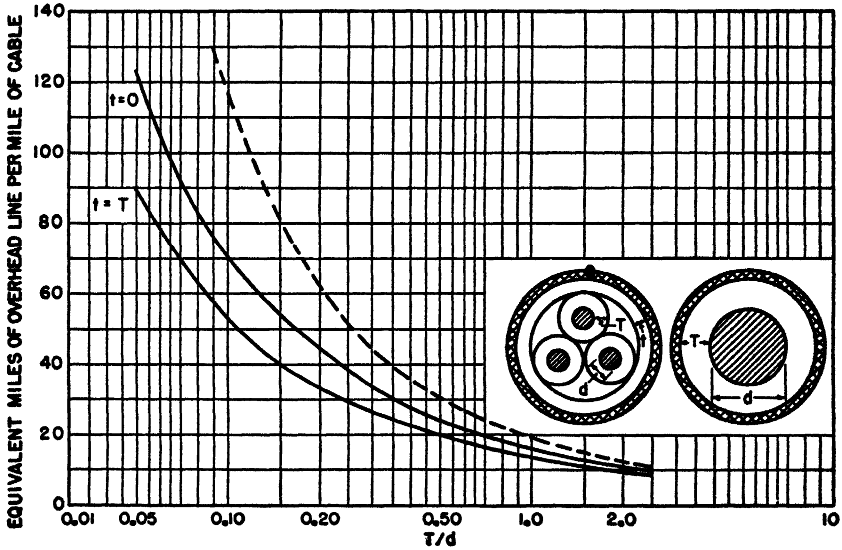

FIG. 12. Overhead line capacitance equivalent of cables. Dashed curve, single-conductor; solid curves, three-conductor. For paper insulated cables with a dielectric constant of 3.8. For varnished cambric or rubber insulation, multiply the equivalent miles by 1.35. Overhead line equivalent based on positive-sequence capacity susceptance of 6.0 micromhos per mile.

Figure 11 can be used to interpret the first criterion in terms of total transformer kilovolt-amperes, system voltage, and maximum allowable lengths of line on the transformer side of the open conductor location.

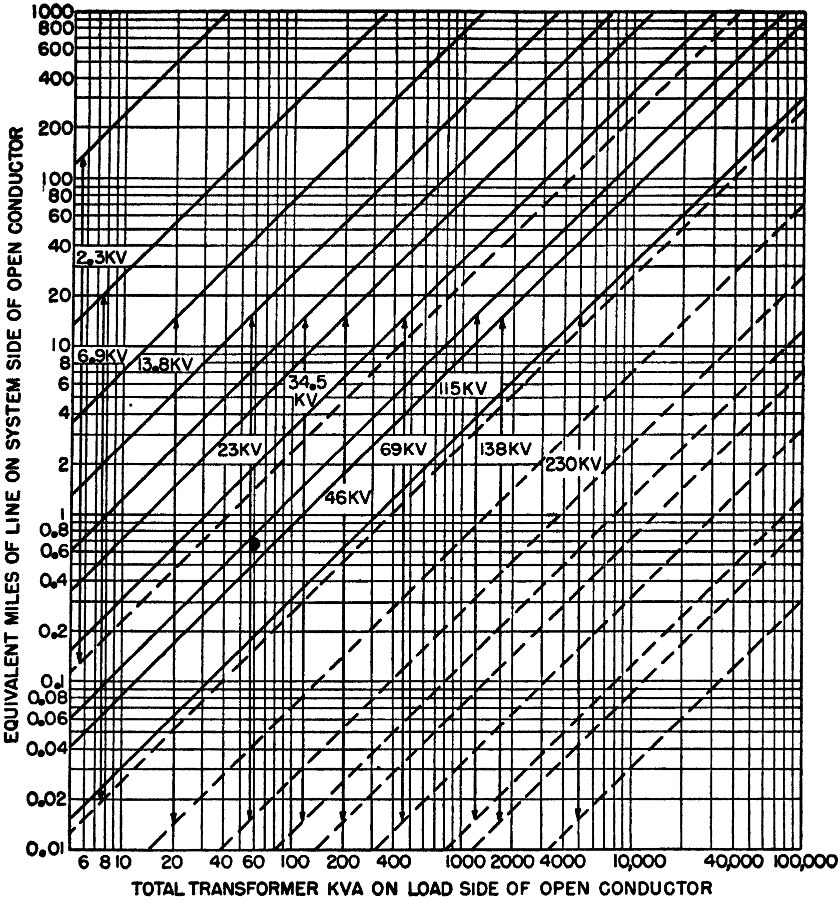

Fig. 13. Isolated neutral system; approximate range of equivalent line lengths on system side of fuse or single-pole switch location which may cause voltages greater than $\sqrt{3}$ times normal with one conductor open and a single line-to-ground fault on the load side. $x'_{c0}/x_m$ must be greater than 40.0 or less than 0.04 if $I_m = 3.5\%$ rms average exciting current. *Notes:* (1) Allowable length of line varies directly with rms average exciting current. (2) This criterion alone is not sufficient for elimination of overvoltages. The criterion of Fig. 11 must be met also. Maximum permissible miles of line in Fig. 11 should be divided by 4.0 for the isolated neutral system.

This figure is based on a ratio of $x_{c1}/x_m = 6.0$ and, since for the isolated system this must be increased to approximately 25.0, it is necessary to divide the miles of line shown by approximately 4.0 in order to conform to the required criterion.

In Fig. 13 the second criterion, imposing the restrictions on the lengths of line on the system side of the open conductor, is shown.

This curve shows that overvoltages may be expected as a result of faults and subsequent fuse blowing for wide ranges of system constants.

In applying these criteria, it should be pointed out that these are approximate values. In some cases it may be necessary to make a more detailed calculation. Exact line configuration, exact magnetizing reactance, and more detailed knowledge concerning the particular transformer saturation characteristics may be necessary. However, the curves serve to indicate in general the range of system constants which can give rise to high overvoltages in isolated neutral systems. Under lightly loaded conditions, it would seem that a relatively high percentage of isolated neutral systems are subject to overvoltages as a result of fuse blowing or single-pole switching under permanent fault conditions.

Other combinations of faults and open conductors are possible in isolated systems which might tend to shift somewhat the range of constants causing high voltages. In distribution circuits involving single-phase feeders, a single blown fuse can sometimes cause overvoltages if the system constants are of certain values. Generally, such cases can be interpreted in terms of three-phase constants such that one of the curves given here can be used in determining the overvoltages possible.

*The effect of load on the transformers* is to reduce the voltages on the open phases obtained with the transformers unloaded.

**Capacitors.** Although capacitance in the above discussion is that inherent in transmission circuits, the effects of capacitors can be taken into account by combining their positive- and zero-sequence capacitances with those of the transmission circuit. If capacitors are Y-connected and grounded, their positive- and zero-sequence capacitances are equal; if ungrounded, their zero-sequence capacitances are zero.

## Phase Reversal

With the transformers loaded and phase *a* open, $V_a$, the voltage of the open phase on the transformer side of the opening referred to the voltage on the system side, will have a quadrature component in addition to the positive or negative in-phase component. If the in-phase component is negative and greater than $-0.5$, $V_{a2}$ will be greater than $V_{a1}$, and there will be phase reversal. Running motors, however, will not slow down if the difference between the positive and negative sequence torques is equal to that required to maintain rotation in the positive direction; but if it is less, they will slow down and stop. They will then reverse their direction of rotation if the negative-sequence torque is great enough to overcome the positive-sequence torque and provide the torque required to start the motors from rest. With a motor running in the positive direction, the positive sequence torque is greater than the negative for equal positive- and negative-sequence voltages. At standstill, these torques are equal for equal voltages. (See Chapter VI.)

The conditions under which loaded motors will reverse their direction of rotation will depend upon the characteristics of the motors, their loads, and the relative magnitudes of the positive- and negative-sequence voltages at their terminals, the latter being influenced by the presence of the motors and other loads on the transformers. For the unloaded transformer bank, the highest ratio of $V_{a2}$ to $V_{a1}$ occurs for $V_a = -2$. For this case,

$$
\begin{aligned}
V_{a1} &= \frac{1}{3}\left[V_a - \frac{1}{2}(V_b + V_c) + j\frac{\sqrt{3}}{2}(V_b - V_c)\right] \\
&= \frac{1}{3}\left[-2 + \frac{1}{2} + j\frac{\sqrt{3}}{2}(-j\sqrt{3})\right] = 0 \\
V_{a2} &= \frac{1}{3}\left[V_a - \frac{1}{2}(V_b + V_c) - j\frac{\sqrt{3}}{2}(V_b - V_c)\right] \\
&= \frac{1}{3}\left[-2 + \frac{1}{2} - j\frac{\sqrt{3}}{2}(-j\sqrt{3})\right] = -1
\end{aligned}
$$

Loads on the transformer bank will modify this ratio, but if $V_{a1}$ is small relative to $V_{a2}$ and $V_{a2}$ is high (unity or above), reversal of direction of rotation of motors (observed in the field and reproduced in the laboratory) will occur.

Phase reversal is further discussed in Chapter VI.

## CALCULATIONS

The voltages arising from one or two open conductors, with or without ground faults, in a circuit supplying an unloaded ungrounded transformer bank can be calculated with reasonable accuracy by consideration only of fundamental-frequency components of voltage and current. As transformer losses are neglected and harmonics are not taken into account, a perfect check between calculations and tests is not to be expected. However, in this study it was through calculations that the order of magnitude of the abnormal voltages and the region in which they occur was determined, and also that three different values of sustained voltage are possible for certain values of $x_{c1}/x_m$ Even with all the calculating devices available at present, it is sometimes necessary and often helpful to make a few preliminary long-hand calculations in the initial stages of an investigation. Also, there are occasions, times, and places where the required calculating devices are not available. For this reason, the method of calculation used here is given in detail for one case and the other cases are outlined so that they may be applied to similar problems where the following assumptions can be made:

1. The sustained voltages to be determined are substantially sinusoidal voltages of fundamental frequency.
2. The effective exciting reactance of transformer windings to be used in calculating fundamental-frequency voltages can be determined from the saturation curve of the transformer, determined by applying sinusoidal voltages. The exciting reactance varies with the magnitude of the impressed voltage, but at any voltage it is the ratio of the rms voltage to the rms current.
3. Resistance in the transformers as well as in the transmission circuit can be neglected.

Transformers may be connected Δ-Y, Δ-Δ, Y-Δ, or Y-Y. Let

$V_N$ = voltage to ground of neutral of Y-connected primary windings in per unit of normal line-to-neutral voltage

$x_m$ = per unit effective transformer exciting reactance of any winding at normal voltage, either *Y*- or Δ-connected

$x_{ab}, x_{ac}, x_{bc}$ = per unit effective transformer exciting reactances of windings *ab*, *ac*, *bc* of Δ-connected bank corresponding to voltages across them

$x_a, x_b, x_c$ = per unit effective transformer exciting reactances of windings *a*, *b*, *c* of Y-connected bank corresponding to voltages across them

$x'_{c1}, x'_{c0}, x_{c1}, x_{c0}$ = per unit positive- and zero-sequence system and circuit line capacitive reactances, respectively, corresponding to $C'_1$, $C'_0$, $C_1$, and $C_0$ of Fig. 3(*b*), based on normal line-to-neutral voltage in the transformer

All reactances are in per unit on a common kva base. Effective reactances of Δ-connected transformer windings, as defined above, are expressed in per unit on normal line-to-line voltage. They are to be multiplied by 3 to be expressed on normal line-to-neutral voltage, which is the base voltage used in calculations.

### Case I. Power Source Solidly Grounded

**A. Phase *a* Open. No Fault.** This case is shown in Fig. 5(*a*) for a Δ-Y-connected transformer bank. No voltage is applied to phase *a* of the three-phase circuit supplying the transformer bank. The voltages to ground, $V_b$ and $V_c$, applied to phases *b* and *c* in per unit of normal line-to-neutral voltage, with the applied voltage of phase *a* on the system side as reference vector, are

$$V_b = -\tfrac{1}{2} - j\frac{\sqrt{3}}{2}; \qquad V_c = -\tfrac{1}{2} + j\frac{\sqrt{3}}{2}$$

Let $V_a$ = voltage to ground of phase *a* on transformer side of opening in per unit of normal line-to-neutral voltage.

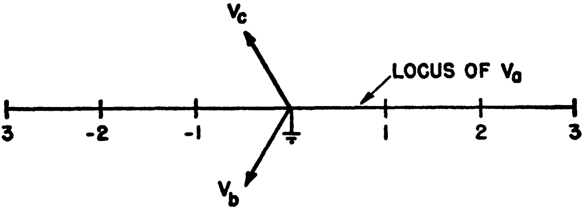

FIG. 14. Locus of vector voltage $V_a$ with transformer resistance neglected. Solidly grounded system, one conductor open.

$V_b$ and $V_c$ can be resolved into two components of voltage in quadrature with each other, $-\frac{1}{2}$ and $-j\sqrt{3}$. The former sends current to ground through phases *b* and *c* in parallel in series with phase *a*. The latter produces equal and opposite currents in phases *b* and *c*. The voltages across windings *ab* and *ac* of Δ-connected windings will be equal in magnitude although not in phase, because the components of voltage across them due to the applied voltage $-j\sqrt{3}$ are $-j\sqrt{3}/2$ and $j\sqrt{3}/2$ while the components due to $-\frac{1}{2}$ are equal and, with resistance neglected, are in quadrature with the *j* components. With $|V_{ab}| = |V_{ac}|$, $X_{ab}$ and $X_{ac}$, which depend upon the *magnitudes* of the voltages $V_{ab}$ and $V_{ac}$, will be equal. Likewise, the voltages across windings *Nb* and *Nc* of Y-connected windings will be equal in magnitude, and $x_b$ and $x_c$ will be equal. The component $-j\sqrt{3}$ produces no voltage at terminal *a* of Δ-connected windings, and no voltage at the neutral *N* of Y-connected windings.

The voltage in phase $a$ is due to the component $-\frac{1}{2}$. As resistance is neglected, $V_a$ may be positive or negative but will have no quadrature component. The locus of $V_a$ is indicated by the horizontal line in Fig. 14, where the voltage of phase $a$ on the system side of the opening is reference vector. There are two values for $V_a$ which give normal voltages across transformer windings: $V_a = 1$ and $V_a = -2$. With $V_a = 1$, the voltages are balanced voltages of positive-sequence phase order; for $V_a = -2$, they are balanced voltages of negative-sequence phase order. Let

$I_a$ = total current flowing from phase $a$ on the transformer side of the opening to ground

$I_t$ = current flowing from transformer bank terminal $a$

$x_t$ = effective exciting reactance of the transformer bank to $I_t$

The flow of current produced by the voltage to ground $-\frac{1}{2}$ applied to phases $b$ and $c$ in parallel is indicated by arrows for Δ-connected primary windings in Fig. 5($a$).

The capacitive reactance of the three-phase shunt, $C_1 - C_0$, which is in parallel with $x_t$, is $\frac{3}{2}x_{c0}x_{c1}/(x_{c0} - x_{c1})$. The two parallel paths are in series with the capacitance $C_0$ between phase $a$ and ground.

$$I_a = \frac{-0.5}{-jx_{c0} + \dfrac{j3x_tx_{c1}x_{c0}}{3x_{c1}x_{c0} - 2x_t(x_{c0} - x_{c1})}} \qquad [14]$$

$$V_a = -jx_{c0}I_a = \frac{3x_{c0}x_{c1} - 2x_t(x_{c0} - x_{c1})}{2x_t(x_{c1} + 2x_{c0}) - 6x_{c0}x_{c1}} \qquad [15]$$

**Windings Δ-Connected on Line Side, Y- or Δ-Connected on Load Side.** The reactance $x_t$ of the bank in either case is that of the equal reactances $x_{ab}$ and $x_{ac}$ in parallel; see Fig. 5($b$). Based on line-to-neutral voltage,

$$x_t = \frac{3}{2}x_{ab} \qquad [16]$$

When $x_t$ in [15] is replaced by $\frac{3}{2}x_{ab}$ and numerator and denominator are divided by $3x_{ab}x_{c1}$, the equation becomes

$$V_a = \frac{\dfrac{x_{c1}}{x_{ab}}\dfrac{x_{c0}}{x_{c1}} - \left(\dfrac{x_{c0}}{x_{c1}} - 1\right)}{1 + 2\dfrac{x_{c0}}{x_{c1}} - 2\dfrac{x_{c1}}{x_{ab}}\dfrac{x_{c0}}{x_{c1}}} \qquad [17]$$

If $x_{c0} = x_{c1}$,

$$V_a = \frac{\dfrac{x_{c1}}{x_{ab}}}{3 - 2\left(\dfrac{x_{c1}}{x_{ab}}\right)} \quad \text{or} \quad \frac{x_{c1}}{x_{ab}} = \frac{3V_a}{2V_a + 1} \tag{18}$$

If $x_{c0} = 2x_{c1}$,

$$V_a = \frac{2\left(\dfrac{x_{c1}}{x_{ab}}\right) - 1}{5 - 4\left(\dfrac{x_{c1}}{x_{ab}}\right)} \quad \text{or} \quad \frac{x_{c1}}{x_{ab}} = \frac{5V_a + 1}{4V_a + 2} \tag{19}$$

*Saturation Neglected.* If transformer saturation is neglected $x_t = \frac{3}{2}x_m$ for all transformer connections. Then, with $x_{c0} = x_{c1}$ in [15],

$$V_a = \frac{\dfrac{x_{c1}}{x_m}}{3 - 2\dfrac{x_{c1}}{x_m}}$$

With $x_{c0} = 2x_{c1}$ in [15],

$$V_a = \frac{2\dfrac{x_{c1}}{x_m} - 1}{5 - 4\dfrac{x_{c1}}{x_m}}$$

$V_a$ calculated from the above equations is plotted in curves 3 of Figs. 6(*a*) and 6(*b*), respectively.

*Saturation Included.* For any given ratio of $x_{c0}$ to $x_{c1}$, $V_a$ can be expressed in terms of $x_{c1}/x_{ab}$. But, because of saturation in the transformer, $x_{ab}$ cannot be determined until $|V_{ab}|$ is known, and $|V_{ab}|$ depends upon $V_a$, which is the quantity sought. For a transformer bank with a given excitation curve, and a given ratio of $x_{c0}$ to $x_{c1}$, a graph of $V_a$ versus $x_{c1}/x_m$ is obtained as follows:

1. Assume a value for $V_a$($-0.5$, 1, and $-2$ will give three points on the curve which are readily calculated).
2. Substitute the given ratio $x_{c0}/x_{c1}$ in [17] to obtain an equation similar to [18] or [19].
3. From this equation, or from [18] or [19], calculate $x_{c1}/x_{ab}$.
4. Calculate $|V_{ab}|$ in per unit of normal line-to-line voltage from the

equation

$$|V_{ab}| = \frac{1}{\sqrt{3}}|V_b - V_a| = \frac{1}{\sqrt{3}}\left|(-\tfrac{1}{2} - V_a) - j\frac{\sqrt{3}}{2}\right|$$

$$= \frac{1}{\sqrt{3}}\sqrt{(V_a + 0.5)^2 + \tfrac{3}{4}} \qquad [20]$$

5. From the given excitation curve, read the exciting current $I_{ab}$ in per unit of normal exciting current $I_{ex}$ corresponding to $|V_{ab}|$; and determine $x_{ab}/x_m$, where $x_m$, the reactance at normal voltage, is the ratio of normal voltage to normal exciting current. $x_m = 1/I_{ex}$.

6. $x_{c1}/x_m$ corresponding to the assumed value of $V_a$ is the product of $x_{c1}/x_{ab}$ and $x_{ab}/x_m$.

To illustrate the procedure, a point on curve 2 of Fig. 6(*b*) will be calculated by following the 6 steps of the procedure given above and indicating each step by number. For this curve $C_1 = 2C_0$, and the saturation curve is curve 2 of Fig. 4, which is given to larger scale by Fig. 2, Chapter IV.

1. Assume $V_a = -3.0$
2. Equation [19] applies for this case.
3. From [19],

$$\frac{x_{c1}}{x_{ab}} = \frac{5V_a + 1}{4V_a + 2} = 1.4$$

4. From [20],

$$|V_{ab}| = \frac{1}{\sqrt{3}}\sqrt{(2.5)^2 + \tfrac{3}{4}} = 1.528$$

5. From curve 2 of Fig. 4 or from Fig. 2, Chapter IV, $I_{ab}$ corresponding to $|V_{ab}|$ is $14.4I_{ex}$, where $I_{ex}$ is normal exciting current.

$$x_{ab} = \frac{|V_{ab}|}{I_{ab}} = \frac{1.528}{14.4I_{ex}}$$

Since $I_{ex} = \dfrac{1}{x_m}$,

$$\frac{x_{ab}}{x_m} = \frac{1.528}{14.4} = 0.106$$

6. $$\frac{x_{c1}}{x_m} = \frac{x_{c1}}{x_{ab}} \times \frac{x_{ab}}{x_m} = 1.4 \times 0.106 = 0.148$$

Point $V_a = -3$, $x_{c1}/x_m = 0.148$ is plotted in curve 2 of Fig. 6(*b*). The calculations for additional points on this curve are tabulated in Table I.

TABLE I

CALCULATED VALUES OF CURVE 2, FIG. 6(*b*)

(Solidly Grounded System. Phase *a* Open. No Fault. Transformers Δ-Connected on Line Side, Y or Δ on Load Side. Saturation Curve 2 of Fig. 4. $C_1 = 2C_0$.)

| $V_a$ (assumed) | $x_{c1}/x_{ab}$ | $\lvert V_{ab}\rvert$ | $x_{ab}/x_m$ | $x_{c1}/x_m$ |
|---|---|---|---|---|
| 0 | 0.5 | 0.577 | 2.76 | 1.38 |
| 0.1 | 0.625 | 0.608 | 2.77 | 1.72 |
| 0.25 | 0.75 | 0.705 | 2.71 | 2.04 |
| 0.5 | 0.875 | 0.764 | 2.18 | 1.90 |
| 1.0 | 1.00 | 1.00 | 1.00 | 1.00 |
| 2.0 | 1.10 | 1.528 | 0.107 | 0.118 |
| 2.5 | 1.125 | 1.803 | 0.056 | 0.063 |
| 3.0 | 1.143 | 2.08 | 0.036 | 0.041 |
| −0.1 | 0.312 | 0.608 | 3.04 | 0.095 |
| −0.2 | 0 | .. | ... | 0 |
| −0.2 to −0.5 | Negative | Impossible values of $V_a$ | | |
| −0.5 | ∞ | ... | .. | ∞ |
| −0.1 | 2.00 | 0.577 | 2.76 | 5.52 |
| −1.5 | 1.625 | 0.764 | 2.18 | 3.55 |
| −2.0 | 1.50 | 1.00 | 1.00 | 1.50 |
| −3.0 | 1.40 | 1.528 | 0.105 | 0.15 |

**Primary Windings Y-Connected.** With secondary windings also Y-connected, $x_b = x_c$, and

$$x_t = x_a + \frac{x_b}{2} \qquad [21]$$

With Δ-connected secondaries, neglecting leakage reactance, the sum of the three voltages to neutral of the Y must be zero. Therefore

$$V_{aN} = 2\ (\text{real part of } V_{Nb})$$

If $I_\Delta$ is the circulating current in the delta, assumed opposite to $I_t$ in phase *a*,

$$x_a(I_t - I_\Delta) = 2x_b\left(\frac{I_t}{2} + I_\Delta\right)$$

Therefore

$$I_\Delta = \frac{I_t(x_a - x_b)}{(x_a + 2x_b)}$$

The voltage drop through the bank, from $b$ to $a$, is

$$j\left(\frac{I_t}{2}+I_\Delta\right)x_b + j(I_t - I_\Delta)x_a = jI_t\frac{\frac{9}{2}x_a x_b}{x_a+2x_b}$$

$$x_t = \frac{\frac{9}{2}x_a x_b}{x_a+2x_b} \qquad [22]$$

*Y-Connected Primary and Secondary Windings.* The current $I_t$ through the Y-connected bank produces voltage drops between $b$ and $N$, and $N$ and $a$, in the ratio $x_b/2$ to $x_a$. Replacing $x_t$ in [15] by $x_a + x_b/2$ and dividing numerator and denominator by $x_{c1}/x_m$, $V_a$ becomes

$$V_a = \frac{\left(\frac{3x_{c1}}{x_m}\right)\left(\frac{x_{c0}}{x_{c1}}\right) + \left(\frac{2x_a}{x_m}+\frac{x_b}{x_m}\right)\left(1-\frac{x_{c0}}{x_{c1}}\right)}{\left(\frac{2x_a}{x_m}+\frac{x_b}{x_m}\right)\left(1+\frac{2x_{c0}}{x_{c1}}\right) - 6\left(\frac{x_{c1}}{x_m}\right)\left(\frac{x_{c0}}{x_{c1}}\right)} \qquad [23]$$

The real part of

$$V_{Nb} = \pm\sqrt{|V_{Nb}|^2 - \tfrac{3}{4}} \qquad [24]$$

$$\frac{V_{aN}}{\frac{x_a}{x_m}} = \pm\frac{2}{\frac{x_b}{x_m}}\sqrt{|V_{bN}|^2 - \tfrac{3}{4}} \qquad [25]$$

$$V_a = V_b - V_{Nb} - V_{aN} = -\tfrac{1}{2} - V_{Nb}\,\text{real} - V_{aN} \qquad [26]$$

A graph of $V_a$ versus $x_{c1}/x_m$ for a given ratio of $x_{c0}$ to $x_{c1}$ can be obtained by the following procedure:

1. Assume $|V_{Nb}|$, the magnitude of the voltage between transformer terminal $b$ and neutral in per unit of normal. From the given transformer saturation curve, obtain $x_b/x_m$ corresponding to $|V_{Nb}|$.
2. Calculate the real part of $V_{Nb}$ from [24].
3. Obtain the ratio $V_{aN}/(x_a/x_m)$ from [25], and from the given transformer saturation curve find values of $|V_{aN}|$ and $(x_a/x_m)$ which are in this ratio. The sign of $V_{aN}$ is the same as that of the real part of $V_{Nb}$.
4. Calculate $V_a$ from [26].
5. Knowing $V_a$, calculate $x_{c1}/x_m$ from the implicit equation [23], after replacing $x_{c0}/x_{c1}$ by its given value.

*Y-Connected Primary and Δ-Connected Secondary Windings.* This case differs from the preceding in that $x_t = \frac{9}{2} \times x_a x_b/(x_a + 2x_b)$ and

$$V_{aN} = 2V_{Nb}\,\text{real} \qquad [27]$$

The procedure to determine a graph of $V_a$ versus $x_{c1}/x_m$ is similar to that of the preceding case except that [27] is used to determine $V_{aN}$, and $x_a/x_m$ corresponding to $|V_{aN}|$ is obtained from the transformer saturation curve. $V_a$ is calculated from [26]. With $V_a$ determined, the corresponding value of $x_{c1}/x_m$ can be calculated from [15] if $x_t$ is replaced by its value from [22].

**B. Phases *b* and *c* Open. No Fault.** See Fig. 5(*b*). Case B differs from case A in that the only voltage applied to the circuit supplying the transformer bank is $V_a = 1$; and $I_a$, the current from phase *a*, flows to ground through $C_0$ of phases *b* and *c* in parallel. Making these two changes, with $V_a$ as reference vector,

$$V_b = V_c = I_a\left(-\frac{jx_{c0}}{2}\right) = -\frac{3x_{c1}x_{c0} - 2x_t(x_{c0} - x_{c1})}{2x_t(2x_{c1} + x_{c0}) - 3x_{c1}x_{c0}} \qquad [28]$$

(*a*) *Δ-Connected Primary Windings, Δ- or Y-Connected Secondary Windings.* Replacing $x_t$ in [28] by $3x_{ab}/2$,

$$V_b = V_c = -\frac{\left(\dfrac{x_{c1}}{x_{ab}}\right)\left(\dfrac{x_{c0}}{x_{c1}}\right) - \left(\dfrac{x_{c0}}{x_{c1}} - 1\right)}{\left(2 + \dfrac{x_{c0}}{x_{c1}}\right) - \left(\dfrac{x_{c0}}{x_{c1}}\right)\left(\dfrac{x_{c1}}{x_{ab}}\right)} \qquad [29]$$

$$|V_{ba}| = \frac{1}{\sqrt{3}}\left|V_a - V_b\right| = \frac{1}{\sqrt{3}}|1 - V_b| \text{ times normal line-to-line voltage} \qquad [30]$$

Using [29] and [30] instead of [17] and [20], respectively, the procedure for determining a graph of $V_b$ or $V_c$ versus $x_{c1}/x_m$ for a given ratio $x_{c0}/x_{c1}$ is analogous to that for one open conductor.

*Y-Connected Primary Windings. Secondary Windings* (*b*) *Y-Connected,* (*c*) *Δ-Connected.* The procedure for obtaining a graph of $V_b$ or $V_c$ versus $x_{c1}/x_m$ for a known ratio, $x_{c0}/x_{c1}$, is analogous to that used for one open conductor, except that there is no quadrature component of voltage in $V_{Nb}$ and [28] is used instead of [15].

## Case II. Power Source Ungrounded

If the capacitance to ground of the power source with ungrounded neutral is large relative to that of the transmission circuit and bank, the power source can be considered to have its neutral effectively grounded. Case II then becomes case I. When this is not the case, the zero-sequence capacitance of the power source, $C_0'$, in Fig. 1(*b*) must be considered. Under the assumption of balanced line-to-line voltages at $a'b'c'$, the positive-sequence capacitance $C'_1$ has no effect upon the resulting voltages.

The procedure used in case II is analogous to that given for case I. The voltage which sends current to ground is 1.5 times normal line-to-neutral voltage applied in a loop circuit in which the ground can be considered a single point.

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# CHAPTER VI

# INDUCTION MACHINES

A rotating machine consists of a stationary and a rotating member, called respectively the stator and the rotor. Rotating a-c machinery is classified as sychronous or asynchronous, depending upon whether normal rotation is at synchronous speed or at a speed different from synchronous speed. In the most common type of synchronous machine, the winding of one member (the field) carries direct current which produces a magnetic field of constant polarity, while the winding of the other member (the armature) carries alternating current; in asynchronous machines, both the stator and rotor windings carry alternating current. Of asynchronous machines, the most widely used is the induction motor. Practically all a-c electrical power is developed by synchronous generators; only a very few induction generators are used. In the utilization of electrical energy, however, the induction motor is of prime importance.

Synchronous machines are discussed in the following chapter. This chapter is devoted to induction motors, with a brief discussion of the induction generator at the end of the chapter.

## INDUCTION MOTORS

Induction motors are built for single-phase, two-phase, or three-phase power supply and to meet the special requirements of the whole range of possible motor drives. Many types of single-phase motors are required in the appliance field alone.[1] In power system studies, the characteristics of the larger induction motors only are ordinarily required. The smaller induction motors can usually be grouped together and treated as a portion of the system load or combined with other types of load into a composite load, characteristic of a distribution feeder.

**Stator Winding.** With the stator as primary, the number of phases in the stator winding will depend upon the supply circuit from which the motor will operate. The number of poles with but few exceptions varies from two to sixteen, a common number being six. The phase windings are distributed, so that the mmf produced by current in any phase is approximately sinusoidal in space.

**Rotor Winding.** Rotor windings of induction motors are of two general types, coil-wound and squirrel-cage. The windings of coil-wound rotors are designed with the same number of poles as the stator winding, but the number of phases need not necessarily be the same. The phase windings are distributed, and the mmf produced by rotor phase currents is approximately sinusoidal in space. The rotor of a three-phase motor usually has three phases, with the axes of the phases 120 electrical degrees apart in space. The terminals of the phase windings of the coil-wound rotor are brought out to slip rings so that they can be shorted through variable external resistances, which are cut in during starting but cut out for normal operation under load. Connections to the rotor windings made at the slip rings constitute the *rotor external circuit.* The squirrel-cage winding is shorted on itself by short-circuiting rings and is not brought out to slip rings.

The discussion in this chapter is confined to induction machines which can be analyzed by means of fundamental-frequency symmetrical components. This restricts the analysis to (*a*) machines with symmetrical stator and rotor windings, and (*b*) dissymmetries occurring in the stator or rotor external circuits, but not simultaneously. With simultaneous dissymmetries occurring in stator and rotor external circuits, harmonics of appreciable magnitudes may be present so that an analysis by means of fundamental-frequency symmetrical components is inadequate. H. C. Stanley[2] has applied $\alpha$ and $\beta$ components (see Volume I, Chapter X) to induction machines and derived general equations from which the effect of harmonics may be obtained. (See also references 3 and 4.)

**Assumption.** In mathematical analyses of induction motors, the following idealization is usually made:

(*a*) Saturation and hysteresis are neglected, except possibly to introduce them approximately as will be shown later.

(*b*) The self-inductance of any rotor or stator circuit is constant. This assumes a uniform air gap.

(*c*) The mutual inductance between any stator or rotor circuit varies sinusoidally with rotor position. This assumes either that the stator and rotor windings are sinusoidally distributed, in the case of a slip-ring induction motor, or that the flux wave produced by the stator mmf is sinusoidal in space with no harmonics, in the case of a squirrel-cage rotor.

This idealization, ordinarily used in the analysis of induction motor performance, has been found by experience to be adequate except for special problems such as subsynchronous locking torques.

With positive-sequence currents of fundamental frequency only in the stator of a polyphase induction motor, the fundamental space wave of mmf due to these currents rotates forward at synchronous speed relative to the stator. (See Appendix B for proof.) The fundamental space wave of flux due to stator mmf also rotates forward at synchronous speed relative to the stator. Likewise, with negative-sequence currents of fundamental frequency only in the stator, there will be a fundamental space wave of flux which rotates *backwards* at synchronous speed relative to the stator. Under the assumptions given above, there will be no harmonic space waves of flux resulting from either positive- or negative-sequence currents of fundamental frequency in the stator. With both positive- and negative-sequence currents ($I_{a1}$ and $I_{a2}$) in the primary winding, there will be both forward and backward rotating fundamental space waves of flux due to these currents.

**Positive Direction of Rotation and Synchronous Speed.** The direction of rotation of the fundamental mmf space wave, resulting from positive-sequence currents in the primary winding, is the positive direction of rotation of the rotor; the speed of this mmf wave is *synchronous speed.*

Synchronous speed in revolutions per minute is $120f/p$, where $p$ is the number of poles and $f$ the impressed frequency; synchronous speed is independent of the number of phases.

**Slip.** The difference between synchronous speed and rotor speed is the *slip*. Slip is indicated by $s$ and expressed in per cent or per unit of synchronous speed. Under normal operation, the slip of an induction motor is small — between 1% and 5%; usually, the larger the motor, the smaller the slip.

**Normal Operation of Polyphase Induction Motor.** The revolving magnetic field, resulting from positive-sequence currents in the stator of a polyphase induction motor, cuts the rotor inductors and induces voltages in them of magnitude and frequency proportional to the slip. With the rotor circuits closed, currents will flow in them of slip frequency $sf$ relative to the rotor, where $f$ is primary impressed frequency and $s$ is per unit slip. The flux produced by rotor currents of slip frequency $sf$ rotates at slip speed with respect to the rotor, just as the flux produced by positive-sequence stator currents of fundamental frequency rotates at synchronous speed with respect to the stator. As rotation is in the same direction as that of the rotor, the speed of the rotor flux with respect to the stator is synchronous speed. The rotor flux and the stator flux therefore rotate at the same speed, relative to the stator, whatever the slip. The resultant air-gap flux, which is the vector sum of the air-gap components of the fluxes resulting from stator and rotor currents, rotates at synchronous speed relative to the stator. It is possible therefore to construct an equivalent circuit where all quantities are referred to the stator.

The interaction of stator and rotor fluxes, corresponding to their product times the sine of the angle between them, produces a torque acting in a direction depending upon whether the stator flux is leading or lagging the flux produced by the rotor current. In a motor, the air-gap component of flux produced by stator current leads the air-gap component of flux produced by the rotor so that torque acting in the rotor in the direction of rotation is produced. (If the rotor is run mechanically at a speed above synchronous speed, the torque is reversed, and the induction machine acts as an induction generator.) With no load except friction and windage, the rotor speed will approach synchronous speed, and there will be but small induced rotor voltages and currents. As the load is increased, the slip will increase until the current in the rotor circuit is sufficient to supply the required torque.

## Three-Phase Induction Motors

The phase windings of the stator may be connected in either Δ or Y. When connected in Y, the neutral is seldom brought out, and therefore the zero-sequence impedance is infinite. Zero-sequence voltages, however, may exist at the motor terminals and at the neutral of Y-connected windings.

**Positive-Sequence Equivalent Circuit.** The induction motor with blocked rotor on open circuit is similar to a three-phase transformer with open-circuit secondary. With the rotor circuits closed and the rotor blocked, it is similar to a transformer with shorted secondary. With the rotor at standstill, the slip is unity ($s = 1$), and the induced voltage and current in the rotor circuit are of primary impressed frequency. Expressed in per unit, with base voltages in the primary and secondary circuits directly proportional to the equivalent number of turns in the two windings, the equivalent circuit for the motor with $s = 1$ is similar to that of the two-winding transformer given in Fig. 2 of Chapter IV.

At any slip $s$ different from unity, the induced voltage in the rotor circuit is directly proportional to the slip in magnitude, and the frequency of rotor voltages and currents is $sf$, where $f$ is the primary frequency and $s$ is per unit slip. With iron losses neglected, an equivalent circuit *referred to the stator* can be constructed from a consideration of Fig. 1($a$), where the subscripts $p$ and $s$ refer to primary (stator) and secondary (rotor) circuits, respectively; $L$ and $M$ represent self- and mutual inductance, respectively, and $r$ resistance. Under steady-state conditions, with balanced voltages of positive sequence applied to the primary,

$$V_m = V_p - I_p[r_p + j\omega(L_p - M)] \tag{1}$$

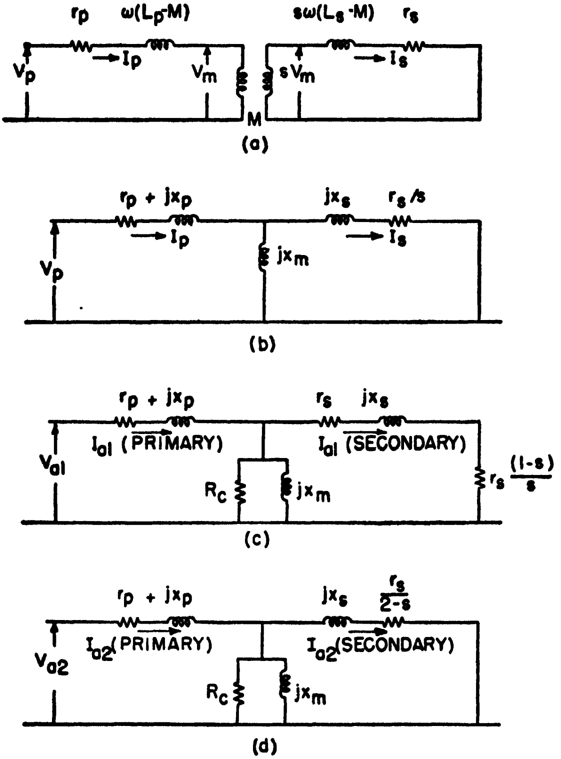

FIG. 1. (*a*) Induction motor diagram with mutual inductance between primary and secondary windings indicated by $M$. (*b*) Equivalent circuit based on [3]. (*c*) Positive-sequence equivalent circuit. (*d*) Negative-sequence equivalent circuit.

where $V_p$ is the positive-sequence applied voltage of the reference phase and $V_m$ is the induced voltage in the primary circuit of the reference phase, corresponding to the air-gap flux which is the vector sum of the fluxes resulting for stator and rotor currents. The induced voltage in the rotor circuit is $sV_m$, and its frequency is $sf$, where $f$ is primary impressed frequency, $s$ is per unit slip, and $\omega = 2\pi f$.

In the secondary circuit under steady operating conditions,

$$sV_m = I_s[r_s + js\omega(L_s - M)] \tag{2}$$

or dividing by $s$,

$$V_m = I_s\left[\frac{r_s}{s} + j\omega(L_s - M)\right] \qquad [3]$$

$$= I_s\left[r_s + j\omega(L_s - M) + r_s\frac{(1-s)}{s}\right] \qquad [3a]$$

Equation [3] refers the induced voltage in the secondary winding to the primary, in both magnitude and frequency. Because the air-gap flux produced by rotor currents rotates at synchronous speed relative to the stator and reacts with the air-gap flux produced by stator current at impressed frequency, [3] expresses the conditions in the secondary circuit as seen from the primary circuit for any slip and resultant frequency in the secondary circuit. It will be noted that the ratio of resistance to reactance in [2] has been preserved in [3].

If, as in the transformer equivalent circuit, self-reactance minus mutual reactance is replaced by the leakage reactances $x_p$ and $x_s$ in the primary and secondary circuits, respectively, the equivalent circuit of Fig. 1(*b*) is obtained from [1] and [3]. If the core loss is assumed to vary approximately as the square of the mutual voltage $V_m$, and [3*a*] instead of [3] is used, the equivalent circuit of Fig. 1(*c*) is obtained in which $R_c$ represents the equivalent core resistance, and $r_s/s$, the equivalent resistance of the rotor, has been replaced by $r_s + r_s(1 - s)/s$. Figure 1(*c*) is the well-known equivalent circuit for one phase of the induction motor under balanced operating conditions (the positive-sequence equivalent circuit), in which all secondary quantities are referred to the primary. It is similar to the equivalent circuit of a two-winding transformer with a resistance load of $R = r_s(1 - s)/s$. The leakage reactances $x_p$ and $x_s$ are generally larger than in a three-phase transformer of the same rating, and the magnetizing reactance $x_m$ is lower, corresponding to a larger magnetizing current. The magnetizing reactance $x_m$ is subject to saturation and decreases with an increase in the voltage $V_m$.

The values of $r_p$, $r_s$, $x_p$, $x_s$, $x_m$, and $R_c$ to be used in the equivalent circuit can be determined by calculation or from tests. The copper resistances $r_p$ of the primary and $r_s$ of the wound-rotor can be measured directly and expressed in per unit, based on the kva rating per phase of the primary winding and base voltages in the primary and secondary circuits which are directly proportional to the equivalent number of turns in the stator and rotor windings, respectively. In a wound-rotor machine, the turn ratio is substantially that of the no-load voltage ratio, with the rotor blocked and its windings open.

With positive-sequence voltage $V_{a1}$ of fundamental frequency applied to the primary, and the rotor blocked with its windings shorted, the sum of the impedances $(r_p + jx_p)$ and $(r_s + jx_s)$ with exciting impedance neglected is obtained from the ratio $V_{a1}/I_{a1}$, where $I_{a1}$ is positive-sequence primary current. From this test, $x_p + x_s$, the sum of the leakage reactances is approximately determined. As in the two-winding transformer equivalent circuit, $x_p$ and $x_s$ are each usually assumed to be equal to one-half the total leakage reactance. With balanced positive-sequence voltages applied and the rotor driven at synchronous speed, there will be no induced rotor voltages or currents in the rotor winding; the self-impedance $Z_{pp}$ of the primary circuit is $V_{a1}/I_{a1}$. When $r_p + jx_p$ and $Z_{pp}$ are known, the core loss resistance $R_c$ and the magnetizing reactance $x_m$ can be obtained. By varying the magnitude of the applied voltage $V_{a1}$, the open-circuit saturation curve is determined. The effect of saturation can be approximately taken into account by adjusting the value of $x_m$ to correspond to the degree of saturation in the motor. The saturation curve with the motor running at no load (except friction and windage) will differ slightly from the synchronously driven saturation curve, as there is a small slip at no load.

The positive-sequence impedance of the induction motor under balanced steady-state conditions can be obtained from Fig. 1 (*c*). It is $Z_1 = V_{a1}/I_{a1}$, where $V_{a1}$ and $I_{a1}$ are the voltage and current, respectively, of phase *a* at the primary terminals. A change in slip will result in a different value of $Z_1$. When it is known that the slip is constant, a constant value of $Z_1$ corresponding to known power input, terminal voltage, and power factor can be used for $Z_1$.

If $V_{a1}$ and $I_{a1}$ are expressed in volts and amperes, respectively, the power input to the motor in watts is $3\,|V_{a1}||I_{a1}|\cos\theta_1$. With $V_{a1}$ and $I_{a1}$ in per unit, the power input in per unit of three-phase base power, taken numerically equal to three-phase base kva, is

$$\textit{Power input} = |V_{a1}||I_{a1}|\cos\theta_1 \qquad [4]$$

where $\cos\theta_1$ is the power factor at the primary terminals.

**Rating.** The rating of an induction motor is normally given in terms of ***horsepower output.*** The horsepower output multiplied by 0.7457 gives the kilowatt output. The kilowatt input is the output plus motor losses; it is also the kilowatt output divided by the per unit efficiency. The ***kva input*** at rated voltage and current is the kilowatt input divided by the power factor at the motor terminals.

The per unit power transferred across the air gap is the power input minus the stator and core losses. It is also the power in the equivalent circuit corresponding to the positive-sequence secondary current ($I_{a1-sec}$) flowing through the equivalent secondary resistance $r_s/s$.

$$\textit{Air-gap power} = |I_{a1-sec}|^2 \frac{r_s}{s} \qquad [5]$$

$$= \text{power input} - (\text{stator copper losses} + \text{core losses})\cdot$$

In [5], the core losses for both stator and rotor have been taken on the primary side.

The positive-sequence air-gap power in per unit is also the positive-sequence torque $T_1$ in per unit developed by the induction motor, where base torque is torque at base kw and base (synchronous) speed.

The per unit positive-sequence torque $T_1$ is

$$T_1 = |I_{a1-sec}|^2 \frac{r_s}{s} \qquad [6]$$

$T_1$ in [6] is also the per unit power which the motor would develop if it were running at synchronous speed. As the speed of the rotor is $1 - s$,

$$\begin{aligned} \textit{Shaft output power} &= \text{torque} \times \text{speed} = T_1\ (1 - s) \\ &= |I_{a1-sec}|^2 r_s \frac{1 - s}{s} \end{aligned} \qquad [7]$$

The actual per unit rotor resistance is $r_s$; the per unit rotor current is $I_{a1-sec}$; therefore, in per unit,

$$\textit{Rotor copper loss} = |I_{a1-sec}|^2 r_s \qquad [8]$$

The shaft output power can also be obtained by subtracting [8] from [5].

In the equivalent circuit of Fig. 1($c$) the *output power* is represented by the power taken by the resistance $[r_s(1 - s)/s]$, the primary and secondary copper losses by the power loss in $r_p$ and $r_s$, and the core losses by the power loss in the resistance branch $R_c$ of the exciting impedance.

**Negative-Sequence Equivalent Circuit.** As the slip $s$ is defined in terms of the speed of the rotor in the positive direction with respect to the speed of the flux resulting from positive-sequence primary currents, the slip with respect to the flux resulting from negative-sequence primary currents, when the rotor is rotating in the positive direction, is $2 - s$. If $s$ in Fig. 1($b$) is replaced by $2 - s$, the negative-sequence equivalent circuit of Fig. 1($d$) is obtained.

From Fig. 1($d$), or by replacing $s$ in [5]–[8] by $2 - s$, the following equations are obtained.

The per unit negative-sequence power across the air gap is

$$\textit{Air-gap power} = |I_{a2-sec}|^2 \frac{r_s}{2-s} \qquad [9]$$

The per unit negative-sequence torque $T_2$ developed by the motor is

$$T_2 = -|I_{a2-sec}|^2 \frac{r_s}{2-s} \qquad [10]$$

where the minus sign indicates that the direction of the negative-sequence torque $T_2$ is opposite to the direction of the positive-sequence torque $T_1$, positive direction corresponding to that of $T_1$. The *air-gap power* given by [9], however, is positive.

The negative-sequence secondary copper loss is

$$\textit{Rotor copper loss} = |I_{a2-sec}|^2 r_s \qquad [11]$$

Subtraction of [11] from [9], or multiplication of [10] by $1 - s$ (the speed of the rotor in the positive direction) gives

$$\textit{Shaft output power} = -|I_{a2-sec}|^2 r_s \frac{(1-s)}{2-s} \qquad [12]$$

where the negative sign in [12] indicates that the shaft output power resulting from negative-sequence currents, when the direction of rotation of the rotor is positive, acts to reduce the output power resulting from positive-sequence currents.

If *negative-sequence voltages only* are applied at the motor terminals, only negative-sequence currents will flow in the primary and secondary windings, and the torque will be negative, tending to rotate the rotor in the negative direction. With the torque and direction of rotation both negative, the shaft output power will be positive. Since the motor is symmetrical, it will operate as well in the negative as in the positive direction as far as its electrical characteristics are concerned.

If *unbalanced voltages*, with both positive- and negative-sequence components, are applied to the motor terminals, the positive- and negative-sequence torques will be in opposite directions. The torque developed by the motor in the positive direction, neglecting the second harmonic torque pulsation, is obtained by adding [6] and [10]:

$$T_1 + T_2 = |I_{a1-sec}|^2 \frac{r_s}{s} - |I_{a2-sec}|^2 \frac{r_s}{2-s} \qquad [13]$$

Multiplication of [13] by $1 - s$ (the speed of the rotor in the positive direction) or addition of [7] and [12] gives

$$\textit{Shaft output power} = |I_{a1-sec}|^2 \frac{r_s(1-s)}{s} - |I_{a2-sec}|^2 \frac{r_s(1-s)}{2-s} \qquad [14]$$

$$\textit{Rotor copper losses} = |I_{a1-sec}|^2 r_s + |I_{a2-sec}|^2 r_s \qquad [15]$$

Addition of [14] and [15] or of [5] and [9] gives the power across the air gap.

$$\textit{Air-gap power} = |I_{a1-sec}|^2 \frac{r_s}{s} + |I_{a2-sec}|^2 \frac{r_s}{2-s} \qquad [16]$$

A comparison of [16] and [13] shows that with both positive- and negative-sequence currents in the stator winding, the power across the air gap is not equal in per unit to the torque developed by the induction motor. Since the positive- and negative-sequence torques are in opposite directions, the negative-sequence torque acts to reduce the power output of the shaft, at the same time increasing the losses and slip. Unbalanced voltages are therefore undesirable in the operation of three-phase induction motors.

Although the equivalent circuits of Figs. 1(*c*) and 1(*d*) can be applied to determine the performance of the induction motor with only positive- or only negative-sequence voltages, respectively, applied to the primary terminals, when both positive- and negative-sequence voltages are present the iron losses may increase appreciably, and cannot be properly represented by equivalent circuits having constant impedance elements. The method described here may be used to calculate the copper losses, but refinement beyond the scope of this discussion would be required to determine accurately the total iron losses. The positive- and negative-sequence equivalent circuits, however, are usually adequate for use in calculations of system performance during unbalanced conditions where fundamental-frequency components of currents and voltages only need be considered. At any given slip, the positive- and negative-sequence equivalent circuits of Figs. 1(*c*) and 1(*d*) can be constructed when the motor constants are known.

**Approximate Motor Constants.** Table I gives constants of typical induction motors in per cent based on the kva input and voltage rating of the motor. The slip and terminal power factor at normal voltage and full load are also given. Table I may be used, in cases where actual constants are not given, as a fair approximation for induction motors of the usual speeds.

Because of the relative high magnetizing current, the power factor is higher at full load than at light load. A motor which has a power factor of 90% at full load may have a power factor of 70% or less when lightly loaded.

*Approximate positive- and negative-sequence impedances* of induction motors are useful in system studies where the performance of the motors themselves are not under consideration, and also as a first approximation in the study of induction motor performance.

The *per unit positive-sequence impedance* of a motor, based on rated kva input and terminal voltage, is $V_{a1}/I_{a1}$, where $V_{a1}$ is per unit positive-sequence terminal voltage and $I_{a1}$ at the operating power factor is per unit positive-sequence primary current. If it is known that a motor (or a group of motors) is operated at rated voltage, one-half rated current, and 60% power factor, the per unit positive-sequence impedance is

$$Z_1 = \frac{V_{a1}}{I_{a1}} = \frac{1.0}{0.5\underline{/-\cos^{-1} 0.6}} = 2.0\underline{/53.1^\circ} = 1.2 + j1.6$$

TABLE I

APPROXIMATE CHARACTERISTICS OF THREE-PHASE INDUCTION MOTORS* OF USUAL SPEEDS

| Rating | Full-Load Efficiency | Full-Load Power Factor | Full-Load Slip | Reactances and Resistances in Per Cent Based on Full-Load Kva and Rated Voltage | | | |
|---|---|---|---|---|---|---|---|
| Hp | % | % | % | $x_p + x_s$ | $x_m$ | $r_p$ | $r_s$ |
| Up to 5 | 75–82 | 75–85 | 3–5 | 10–14 | 160–220 | 4–6 | 4–6 |
| 5 to 25 | 80–88 | 82–90 | 2½–4 | 12–16 | 200–280 | 3½–5 | 3½–5 |
| 25 to 200 | 86–92 | 84–91 | 2–3 | 15–17 | 220–320 | 3–4 | 3–4 |
| 200to1000 | 91–93 | 85–92 | 1½–2½ | 15–17 | 240–360 | 2½–3 | 2–3 |
| 1000 up | 93–94 | 88–93 | 1–2 | 15–17 | 260–400 | 1½–2 | 1½–2½ |

Core loss current 2%, approximately.

* Table furnished by P. L. Alger. See also *Electrical Engineers' Handbook*, Electric Power Volume, Harold Pender, Editor-in-Chief, Fourth Edition, John Wiley and Sons, 1949, and *Standard Handbook for Electrical Engineers*, Eighth Edition, McGraw-Hill Book Co.

The *approximate negative-sequence impedance* can be obtained from Fig. 1(*d*). When the slip $s$ is small, $r_{s/(2-s)}$ is small relative to $x_s$, and may be neglected. The exciting impedance ($R_c$ and $jx_m$ in parallel) is large relative to $jx_s$. If the exciting impedance is assumed to be infinite and $r_p$ is neglected, $Z_2$, the per unit negative-sequence impedance of the induction motor viewed from its terminals, is approximately

$$Z_2 = j(x_p + x_s) \qquad [17]$$

From Table I, the approximate per unit value of $Z_2$ may be taken as $j0.16$, based on rated kva input and voltage.

**Starting Torque.** At standstill the slip is unity, and the rotor speed zero. With $s = 1$ in [6], the per unit starting torque is equal to the per unit copper loss in the rotor circuit given by [8]. At any given applied voltage, the starting torque can be increased to a maximum value by increasing the secondary resistance $r_s$. It can be shown that, by increasing the rotor resistance, the slip at which maximum torque occurs is increased but the value of maximum torque is not appreciably changed.[5] Since losses are increased with an increase in slip, a low secondary resistance with resultant small slip is desirable for efficient running conditions, whereas a high value of $r_s$ may be required for starting. For the wound-rotor motor, secondary resistance can be cut in at starting, thereby increasing the starting torque, and cut out when running to obtain a small slip. For the squirrel-cage rotor, a compromise must be made between a high value of $r_s$ for best starting torque and a low value of $r_s$ for best operating conditions.

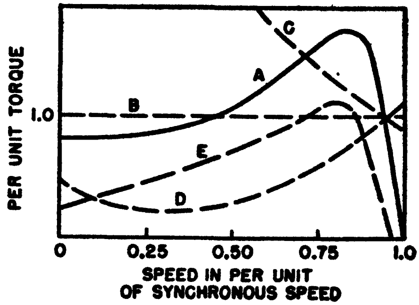

FIG. 2. Speed-torque curves. *A*, motor speed-torque curve at rated voltage. *B*, load speed-torque curve for constant torque. *C*, load speed-torque curve for constant power. *D*, typical load speed-torque curve. *E*, motor speed-torque curve for unbalanced system conditions.

**Motor Speed-Torque Curve.** With normal balanced voltages at the motor terminals, the torque $T_1$ can be calculated from [6], where $I_{a1-sec}$ at any slip $s$ is determined from the positive-sequence equivalent circuit. Curve *A* of Fig. 2 shows the torque-speed curve of an induction motor at rated voltage. The speed of the rotor is $1 - s$ in per unit of synchronous speed. At synchronous speed, $I_{a1-sec}$ is zero, and therefore the torque is zero.

Should a system disturbance occur which reduces the voltages at the motor terminals without unbalancing these voltages, the motor torque at given rotor speed will vary as the square of the motor terminal voltage $V_{a1}$; the torque $T_1$ from [6] varies as $I^2_{a1-sec}$, whereas $I_{a1-sec}$ varies directly as $V_{a1}$. Depending upon the reduction in voltage and the characteristics of the load driven by the motor, the motor may continue to operate or may stall. If equipped with undervoltage relays, the motor will be cut out of service when its terminal voltage drops below a predetermined value.

The operating portion of the motor torque-speed curve is between synchronous speed and maximum torque. Maximum torque is usually around twice the operating torque at rated voltage and full load. At the speed corresponding to maximum torque and at slower speeds, an increase in slip results in a decrease in motor torque, and the motor will slow down and come to rest unless the torque required by the load decreases more rapidly with decreasing speed than the motor torque. The variation of required load torque with rotor speed can be plotted when the character of the load is known.

**Load Speed-Torque Curves.** There are many types of induction motor load. A conveyor, or a weight operated by pulleys, requires approximately constant torque. A motor-driven d-c generator with a voltage regulator delivering power to a fixed impedance load requires constant power during the operating range of the regulator. In the speed-torque curve for a constant power load, the required per unit torque is the reciprocal of per unit speed. The torque required to operate a fan varies approximately as the square of the speed; it, therefore, decreases more rapidly than the decrease in speed. The torque-speed characteristics of many loads driven by induction motors vary through a complete cycle — a mine hoist, for example. Others are subject to sudden changes in required torque, as when an electrically driven dredge strikes rock.

Curves $B$, $C$, and $D$ of Fig. 2 are possible motor-driven load-torque speed curves. Curve $B$ represents a constant-torque load; curve $C$, a constant power load; and curve $D$ a load which requires neither constant torque nor constant power.

## Unbalanced System Conditions

When a system disturbance occurs which produces unbalanced voltages with negative-sequence components at the motor terminals, the positive-sequence torque $T_1$ is reduced if the positive-sequence terminal voltage is below normal. In addition there is a negative-sequence torque $T_2$ acting in the opposite direction to $T_1$. For the motor to continue to rotate, its slip must increase until the positive-sequence torque minus the negative-sequence torque is equal to the torque required by the load at the resultant speed. Just what the slip will be during unbalanced system conditions will depend upon

1. The nature of the unbalance.
2. Normal system operating voltages.
3. System constants including those of the induction motor.
4. The character of the load driven by the induction motor.

The nature of the unbalance determines the connection of the sequence networks to represent the unbalanced condition. (A possible exception is an open conductor. This will be discussed later.) Normal system operating voltages determine the impressed voltages in the positive-sequence network. The constants of the motor determine its positive- and negative-sequence equivalent circuits in terms of the unknown slip. By varying the slip, a torque-speed curve of the motor during the unbalanced system condition can be obtained. With the assumed slip replacing $s$ in the positive- and negative-sequence equivalent circuits of the motor given by Figs. 1($c$) and 1($d$), the sequence networks are connected to satisfy the unbalanced system conditions, and the currents and voltages at any desired system points are calculated. From the positive- and negative-sequence currents, $I_{a1-sec}$ and $I_{a2-sec}$, the total torque developed by the motor is obtained from [13] and plotted versus speed. By assuming several values of slip, a torque-speed curve of the motor for the unbalanced system condition is obtained. Let this torque-speed curve be represented by curve $E$ of Fig. 2. The intersection of curve $E$ with the load torque-speed curve will give the speed of the rotor and its slip. If there is no intersection, or if the intersection is on a part of the curve where the load torque increases more rapidly (or decreases less rapidly) with a decrease in rotor speed than the motor torque of curve $E$, the motor cannot supply the required torque and will stall. In Fig. 2, with a constant power load (curve $C$), the motor is unstable; but with the constant torque load (curve $B$) and the load of curve $D$, the motor is stable. With either curve $B$ or $D$ there are two intersections. Operation at the intersection between synchronous speed and maximum torque on curve $E$ is at a stable operating slip. The other intersections are unstable because an increase in slip is not accompanied by sufficient motor torque to supply the load torque required at the decreased speed. With the slip of the motor determined, the currents and voltages at any desired system point can be obtained.

**Instantaneous Power.** The per unit instantaneous power input to a three-phase induction motor with unbalanced applied voltages can be obtained from Volume I, page 66, [37]. For a motor of infinite zero-sequence impedance, $I_{a0} = 0$, and the equation becomes

$$\begin{aligned} p = {} & |V_{a1}||I_{a1}| \cos \theta_1 + |V_{a2}||I_{a2}| \cos \theta_2 \\ & - |V_{a2}||I_{a1}| \cos (2\omega t + \alpha + \beta - \theta_1) \qquad [18] \\ & - |V_{a1}||I_{a2}| \cos (2\omega t + \alpha + \beta - \theta_2) \end{aligned}$$

Replacement of $|I_{a1}|$ by $\left|\frac{V_{a1}}{Z_1}\right|$ and $|I_{a2}|$ by $\left|\frac{V_{a2}}{Z_2}\right|$ in [18] gives

$$p = \frac{|V_{a1}|^2}{|Z_1|}\cos\theta_1 + \frac{|V_{a2}|^2}{|Z_2|}\cos\theta_2 - \frac{|V_{a1}||V_{a2}|}{|Z_1|}\cos(2\omega t + \alpha + \beta - \theta_1) - \frac{|V_{a1}||V_{a2}|}{|Z_2|}\cos(2\omega t + \alpha + \beta - \theta_2) \qquad [19]$$

where $p$ is instantaneous power in per unit of base three-phase power, numerically equal to base three-phase kva.

$V_{a1}$ and $V_{a2}$ are the positive- and negative-sequence applied voltages in per unit of base voltage per phase.

$\alpha$ and $\beta$ are the angles by which $V_{a1}$ and $V_{a2}$, respectively, lead the reference vector.

$I_{a1}$ and $I_{a2}$ are the positive- and negative-sequence per unit line currents flowing into the motor.

$Z_1$ and $Z_2$ are the positive- and negative-sequence per unit impedances of the motor corresponding to the operating slip.

$\theta_1$ and $\theta_2$ are the impedance angles of $Z_1$ and $Z_2$, the negative signs in [18] and [19] indicating that $I_{a1}$ and $I_{a2}$ lag $V_{a1}$ and $V_{a2}$, respectively.

The first two terms of [18] and [19] give the average power to the motor. The last two terms give the pulsating second harmonic power.

The instantaneous per unit power at the motor terminals is also the per unit air-gap power, neglecting stator losses. The per unit torque developed by the motor can be determined from the per unit air-gap power by reversing the signs of the power terms containing $I_{a2}$. Neglecting stator losses, the per unit rotor torque from [18] or [19] is

$$T = \frac{|V_{a1}|^2}{|Z_1|}\cos\theta_1 - \frac{|V_{a2}|^2}{|Z_2|}\cos\theta_2 - \frac{|V_{a1}||V_{a2}|}{|Z_1|}\cos(2\omega t + \alpha + \beta - \theta_1) + \frac{|V_{a1}||V_{a2}|}{|Z_2|}\cos(2\omega t + \alpha + \beta - \theta_2) \qquad [20]$$

The first two terms of [20], which give the average torque tending to produce rotation, correspond to [13] with stator losses neglected. The last two terms indicate the magnitude of the pulsating second harmonic component of torque due to unbalanced voltages.

It is interesting to note that at standstill, when $s = 1$ and therefore

$|Z_1| = |Z_2|$ and $\theta_1 = \theta_2$, the pulsating power to the motor from [19] is

$$p = -\frac{2|V_{a1}||V_{a2}|}{|Z_1|}\cos\,(2\omega t + \alpha + \beta - \theta_1) \qquad [21]$$

while the pulsating electrical shaft torque on the basis of negligible primary losses from [20] is zero.

If the positive- and negative-sequence applied voltages are equal in magnitude, the torque tending to produce rotation at standstill given by the first two terms of [19] is zero. This condition arises when a single-phase voltage is applied to two terminals of a three-phase motor with the other terminal open. With phase *a* open,

$$I_a = I_{a1} + I_{a2} = 0$$

and, therefore,

$$I_{a1} = -I_{a2}$$

At standstill $Z_1 = Z_2$, and

$$V_{a2} = I_{a2}Z_2 = I_{a2}Z_1 = -I_{a1}Z_1 = -V_{a1}$$

and, therefore,

$$|V_{a2}| = |V_{a1}|$$

With $|V_{a1}| = |V_{a2}|$ in [20], the torque at standstill is zero, and the motor will not start. If it is started, however, or if one phase is opened while it is running under load, the motor will develop torque which will keep it running in the same direction, provided there is a value of operating slip at which the torque required by the load can be supplied by the motor. Although $I_{a1}$ and $I_{a2}$ are equal with the motor running and one phase open, $Z_1$ and $Z_2$ are not equal; $|Z_1|$ is larger than $|Z_2|$ at any operating slip, and $|V_{a1}| = |I_{a1}Z_1|$ is larger than $|V_{a2}| = |I_{a2}Z_2|$. The average torque therefore is in the positive direction, positive direction being the direction of rotation of the rotor.

## Phase Reversal with Open Conductors

Open conductors in circuits which supply ungrounded transformer banks are discussed in the preceding chapter. It is shown that phase reversal, or high sustained voltages to ground on the open phase, or both, will occur for some ratios of line capacitive reactance to normal transformer exciting reactance; but they will not occur for other ratios. It is also shown that for certain ratios of these reactances there are three possible voltage conditions, any one of which may occur, depending upon the phase of the voltage wave at the time that phase is opened. See Chapter V, Fig. 6(*a*), Fig. 6(*b*), and Fig. 7.

Transformer exciting current increases non-linearly with an increase in induced voltage; therefore, when the induced voltages in the three phases are unbalanced with one or two well above normal and the third approximately normal, the transformer bank is not a linear circuit, nor is it a symmetrical circuit. In this case, symmetrical components which are based on linear circuits are not applicable, and some other method must be used in an analytical solution. One such method is given in the preceding chapter. Where the effect of unequal degrees of saturation in the three phases of a transformer bank is unimportant, the bank may be considered a symmetrical linear circuit. This is usually the case in a grounded bank. In general, when power is supplied to induction motors through a grounded transformer bank, and one phase in the supply circuit is open, the method of symmetrical components is applicable. See Volume I, page 144, Fig. 14(*c*), for the connection of the sequence networks to represent the condition of an open conductor in an otherwise *symmetrical* circuit, where linear relations exist.

**Unbalanced Load.** In the following discussion it will be assumed that transformer exciting currents and line capacitances can be neglected. It will also be assumed that the section of the power system involved, except for the open conductor and the unbalanced load, is symmetrical. Under these assumptions, the equations given in Volume I for open conductors and unbalanced loads in terms of symmetrical components and $\alpha\beta 0$ components are applicable.

When positive- and negative-sequence impedances can be assumed equal and zero-sequence currents are absent, the unbalanced loads on the three phases can be represented as constant impedance static loads, and phase quantities can be used in circuit analysis.

An induction motor has unequal positive- and negative-sequence impedances when running. It is therefore only at standstill that it can be classed as a constant impedance static load.

**Constant Impedance Loads.** Let phase $A$ be the open phase. Let the impedances of the loads between phases $C$ and $A$, $A$ and $B$, and $C$ and $B$ be indicated by $Z_{ca}$, $Z_{ab}$, and $Z_{cb}$, respectively, as indicated in Fig. 3, where normal phase order is ABC. In Fig. 3 the load is shown Δ-connected. If Y-connected and ungrounded, it can be replaced by its equivalent Δ. Let the voltage $V_{cb}$ across the load impedance $Z_{cb}$ be reference vector. This voltage will be determined by the voltage of the source, the impedance between the source and load terminals $B$ and $C$, and the vector sum of the load currents $I_{bc}$ and $I$. (See Fig. 3.) But whatever the value of $V_{cb}$, the voltages $V_{ca}$ and $V_{ab}$ can be referred to it in magnitude and phase.

As the load is ungrounded, whether Y- or Δ-connected,

$$V_{cb} = V_{ca} + V_{ab}$$

where $V_{ca}$, $V_{ab}$, and $V_{cb}$ are voltage rises from the terminal indicated by the first subscript to that indicated by the second.

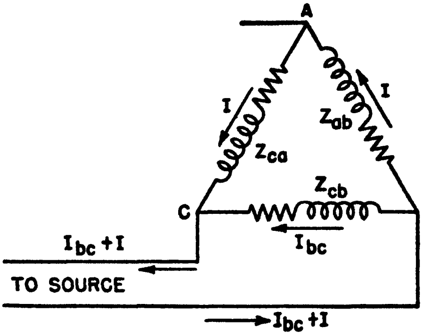

FIG. 3. Open conductor in circuit supplying unbalanced, ungrounded load of constant static impedances.

The same current will flow through impedances $Z_{ca}$ and $Z_{ab}$. Let this current be $I$. Then

$$I = \frac{V_{cb}}{Z_{ca} + Z_{ab}}$$

$$V_{ca} = IZ_{ca} = \frac{V_{cb}Z_{ca}}{Z_{ca} + Z_{ab}} = \frac{V_{cb}}{1 + \dfrac{Z_{ab}}{Z_{ca}}} \qquad [22]$$

$$= V_{cb}\frac{1}{1 + \left|\dfrac{Z_{ab}}{Z_{ca}}\right| \underline{/\theta_{ab} - \theta_{ca}}}$$

where $Z_{ab} = |Z_{ab}|\underline{/\theta_{ab}}$ and $Z_{ca} = |Z_{ca}|\underline{/\theta_{ca}}$.

Equation [22] may be written in the form

$$V_{ca} = V_{cb}\frac{1}{1 + B\underline{/\phi}} \qquad [23]$$

where $B = \left|\dfrac{Z_{ab}}{Z_{ca}}\right|$, $\phi = \theta_{ab} - \theta_{ca}$; and $V_{cb}$ is reference vector.

It is shown in Appendix C that the fraction in [23] is the equation of a circle for any constant value of $B$ and a variable angle $\phi$; it is the arc of a circle for any constant value of $\phi$ and variable $B$. Equations for the centers and radii of these circles and arcs are also given.

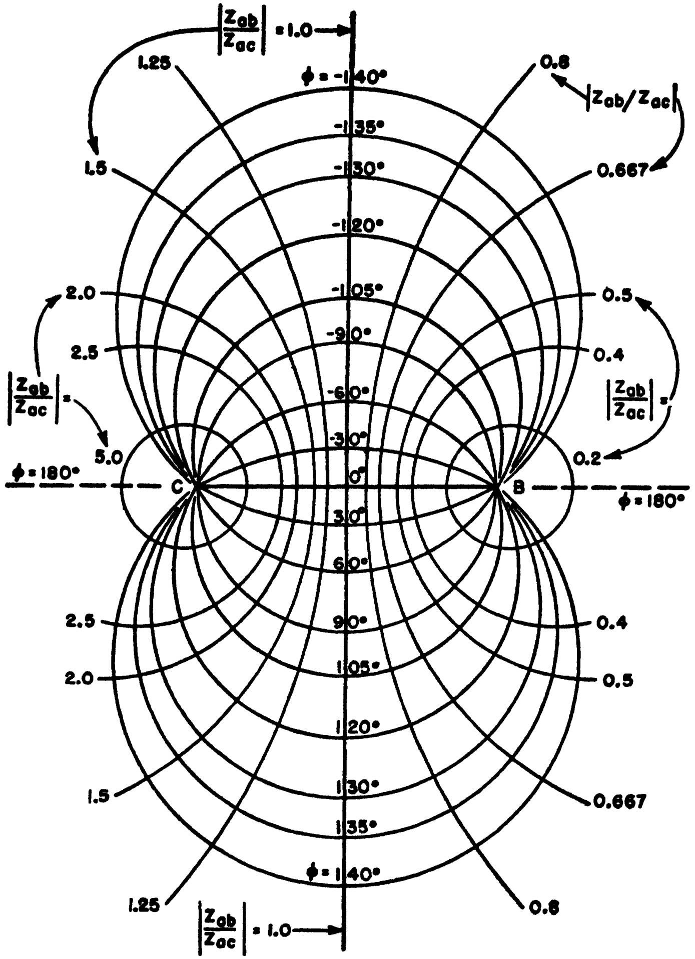

FIG. 4. Graphical solution of [22].

Figure 4 is a graphical solution of [22] for any given values of $|Z_{ab}/Z_{ca}|$ and $\phi = \theta_{ab} - \theta_{ca}$. From this figure, the terminus $A$ of the voltage vector $V_{ca}$ given by [22] can be located at the intersection of a $|Z_{ab}/Z_{ca}|$ circle with a $\phi$ arc, where $\phi = \theta_{ab} - \theta_{ca}$ and $V_{cb}$ is reference vector. With $A$ located, the magnitude and phase of $V_{ca}$ and of $V_{ab}$ relative to $V_{cb}$ are determined. If $A$ lies above $V_{cb}$, $V_{ca}$ leads $V_{cb}$ and the phase order is normal phase order *ABC*; if $A$ lies below $V_{cb}$, the phase order is reversed.

From [22], it may be seen that if $\theta_{ab} - \theta_{ac} = \phi$ is a positive angle ($0 < \phi < 180°$) the $j$ component of the denominator of the fraction is positive, but the $j$ component of the fraction is negative. Therefore $V_{ca}$ lags $V_{cb}$ for positive values of $\phi$ between 0° and 180°, and normal phase order is reversed. If $\phi$ is a negative angle between 0° and $-180°$, $V_{ca}$ leads $V_{cb}$ and the phase order is normal. In the former case, the resultant torque on a motor at standstill is in the negative direction. In the latter case, it is in the positive direction.

If $Z_{ab} = Z_{ca}$, $B = 1$ and $\phi = 0$. From [22], or Fig. 4, $V_{ca} = \frac{1}{2} V_{cb}$ and operation is single-phase.

When the three phases are initially unequally loaded with both three-phase and single-phase loads, the power factors of the three currents will, in general, be different. If all are lagging, the maximum value of $\phi = \theta_{ab} - \theta_{ac}$ will be less than $\pm 90°$. For this condition, phase reversal may occur but, from Fig. 4, it can be seen that neither $V_{ca}$ nor $V_{ab}$ can be greater than $V_{cb}$ in magnitude.

**Capacitors.** When capacitors are used to correct power factors and reduce voltage drops, the condition of unequal load impedances, paralleled by capacitive impedances, may give value of $\phi$ above $\pm 90°$. In this case, $V_{ca}$ or $V_{ab}$, or both, may be greater than $V_{cb}$; and $V_{cb}$ may be higher than normal, if the resultant current flowing from source to load is a leading current flowing through inductive impedance. But phase reversal, which depends upon the sign of $\phi = \theta_{ab} - \theta_{ca}$, may or may not accompany high voltages. If by chance $Z_{ab}/Z_{ca} = 1\underline{/120°}$, the resultant voltage across the three phases will be balanced voltages of negative-sequence phase order. If, on the other hand, $Z_{ab}/Z_{ca} = 1\underline{/-120°}$, the resultant voltages will be balanced positive-sequence voltages.

Further discussion of this subject will be found in references 6–10.

**Single-Phase Motors.** The performance of a single-phase motor with a single-phase distributed stator winding is similar to that of a three-phase motor with one phase open and can be determined in a similar manner. When the single-phase motor is connected between phases $b$ and $c$ of a three-phase circuit, $I_b = -I_c$, and from [2], [3], [10], and [11] of Chapter I,

$$I_{a2} = -I_{a1}; \quad V_{a1} = I_{a1}Z_1; \quad V_{a2} = -I_{a1}Z_2$$

The voltage across the motor terminals is

$$V_{cb} = V_b - V_c = -j\sqrt{3}\,(V_{a1} - V_{a2}) = -j\sqrt{3}\,I_{a1}(Z_1 + Z_2)$$

The current in the motor is

$$I_b = -I_c = a^2 I_{a1} + a I_{a2} = -j\sqrt{3}\, I_{a1}$$

The impedance of the single-phase motor is

$$Z = \frac{V_{cb}}{I_b} = Z_1 + Z_2 \qquad [24]$$

where $Z_1$ and $Z_2$ are determined from the positive- and negative-sequence equivalent circuits of Figs. 1(*c*) and 1(*d*) corresponding to the slip. When expressed in per unit for use in the positive- and negative-sequence networks of the three-phase system, $Z_1$ and $Z_2$ must be based on the system kva per phase and base line-to-neutral voltage.

In a single-phase circuit with a voltage $E$ applied across the motor terminals, the current is

$$I = \frac{E}{Z_1 + Z_2} \qquad [25]$$

where $Z_1$ and $Z_2$, if expressed in per unit, are based on the kva and voltage of the single-phase circuit.

At standstill, the torque tending to turn the rotor is the same in either direction. Single-phase motors therefore require an auxiliary starting device, or an auxiliary winding (sometimes cut out during running conditions) which will produce a resultant rotating flux of such magnitude and direction as to cause rotation in the positive direction from standstill. Because of the pulsating component of torque, single-phase motors are often provided with resilient mountings for the stator and the shaft coupling to reduce the vibration and noise.

When a three-phase motor is operated with two terminals (terminals $b$ and $c$) connected together and a single-phase voltage applied between these terminals and the third terminal (terminal $a$), $V_b = V_c$, and therefore (see Chapter I, [1]–[3]) $V_{a1} = V_{a2}$.

$$I_{a1} = \frac{V_{a1}}{Z_1} \quad \text{and} \quad I_{a2} = \frac{V_{a2}}{Z_2} = \frac{V_{a1}}{Z_2}$$

$$I_a = I_{a1} + I_{a2} = V_{a1}\frac{Z_2 + Z_1}{Z_1 Z_2}$$

$$V_{ba} = V_a - V_b = 2V_{a1} - (a^2 + a)V_{a1} = 3V_{a1}$$

The impedance of the motor between terminals is

$$Z = \frac{V_{ba}}{I_a} = 3\frac{Z_1 Z_2}{Z_1 + Z_2} \qquad [26]$$

In a single-phase circuit with a voltage $E$ applied, the current $I$ is

$$I = \frac{E(Z_1 + Z_2)}{3Z_1Z_2} \tag{27}$$

where $Z_1$ and $Z_2$ correspond to the slip of the motor. $Z_1$ and $Z_2$ of a three-phase motor, if given in per unit, are based on motor rated kva per phase and rated line-to-neutral voltage. For use in a single-phase circuit in which base voltage is $\frac{3}{2}$ times the rated line-to-neutral voltage of the motor and base kva is the three-phase kva rating of the motor, the given values of $Z_1$ and $Z_2$ are multiplied by $(\frac{2}{3})^2 \times 3 = \frac{4}{3}$.

**Short-Circuit Calculations.** The time constant of the rate of decay of the component of rotor flux contributing to short-circuit current is very small compared with that of a synchronous machine. For a terminal fault, the time constant is approximately $(x_p + x_s)/2\pi f r_s$ sec. Because of their comparatively small short-circuit time constants, induction machines are not ordinarily included as sources of short-circuit current in relay or circuit-breaker interrupting duty calculations. However, in determining the approximate magnitude of instantaneous current such as required for fuse and breaker stresses, expulsion tubes, and high-speed (1-cycle) relay studies, their effect as a source of short-circuit current is usually included. This may be done by assuming the induction machine to act during this short interval of time as an equivalent synchronous machine with an internal voltage $E$ and a reactance $x_p + x_s$. The voltage $E$, determined from conditions before the fault, is

$$E = V_p - I_p j(x_p + x_s) \tag{28}$$

where $V_p$ and $I_p$ are the terminal voltage and the current flowing into the reference phase $a$ of the motor before the fault. During the short circuit, the voltage $E$ acts through the reactance $x_p + x_s$ in the positive-sequence network.

When subtransient components of short-circuit currents are determined under the principle of superposition by applying a voltage at the fault equal and opposite to that which existed there before the fault (Volume I, page 26), the induction machine may be represented by its short-circuit reactance which is approximately equal to $x_p + x_s$, just as synchronous machines are represented by their subtransient reactances for determining initial or subtransient components of fault current. The short-circuit current contribution from induction machines is usually not a large part of the total current, and this method, although approximate, is ordinarily sufficiently accurate.

**Induction Motors as Phase Balancers.** Induction motors offer a comparatively low impedance to negative-sequence currents. See [17]. For this reason, they are sometimes used to balance the phase voltages of a system. A large induction motor with a low secondary resistance will have small losses. If connected at a system point where unbalanced voltages are created (for example, by a large single-phase load), it will supply a large part of the negative-sequence current required by the load. This reduces the negative-sequence current in the rest of the system, and phase voltages are less unbalanced.

Turbine generators are usually built with solid rotors, and are not equipped with amortisseur windings unless especially designed to supply single-phase loads. If systems supplied by such generators later develop large single-phase loads, the generators may be damaged by negative-sequence current or require an appreciable reduction in their load-carrying ability, unless they are provided with a low-resistance amortisseur or some form of phase balancer is used to reduce the negative-sequence current in them. (See Volume I, page 259, Problem 2, for method of determining the division of negative-sequence current resulting from a single-phase load.)

## Induction Generator

If the rotor of an induction machine is rotated mechanically at a speed above synchronous speed, the induction machine becomes an induction generator in which the output current leads the terminal voltage. Because of its leading power factor, it can be used only in parallel with synchronous machines which supply the lagging wattless current required by the load and also neutralize the leading wattless current supplied by the induction generator. The terminal voltage and frequency of the induction generator are fixed by its parallel alternator; its slip determines its power output. The leading wattless component of current which it provides depends upon the machine constants. Its power factor is always leading. As the output power increases, the slip and the wattless component of current increase, and the terminal power factor first increases and then decreases.

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# CHAPTER VII

# SYNCHRONOUS MACHINES

The principal source of electric energy in power systems is the synchronous generator. Synchronous condensers are widely used as a means of reducing losses and controlling voltages. Many large loads are driven by synchronous motors. The very great importance of synchronous machines in the operation of a-c power systems has necessitated thorough analyses of their characteristics. Extensive investigations have been made of the influence of these characteristics upon the performance, not only of the machines themselves, but also of the power system as a whole during both normal and abnormal operating conditions. Many excellent papers in the literature have presented the results of analytic study, and of field and factory tests. A few of these, used directly as references, are listed in the bibliography at the end of this chapter. As a result of standardization of notation and availability of technical information, machines with characteristics best suited for given conditions of operation may be specified by operating engineers and built by manufacturers.

In this chapter, the characteristic constants (or parameters, as they are seldom strictly constant) of synchronous machines are defined; and methods of representing synchronous machines in system studies in terms of their characteristic constants are given, with special attention to the determination of fundamental-frequency currents and voltages by the method of symmetrical components. In Volume I, only the simplest equivalent circuits are used to replace synchronous machines in the sequence networks. In the positive-sequence network, the equivalent circuit consists of a voltage $E$ acting through an impedance $Z_1$, the voltage and impedance corresponding to subtransient, transient, or steady-state conditions. In the negative- and zero-sequence networks, the equivalent circuits consist of simple self-impedances $Z_2$ and $Z_0$, respectively. It will be shown that these simple equivalent circuits are satisfactory in many system studies where fundamental-frequency currents and voltages only need be considered.

## Mechanical Features

A synchronous machine may have a stationary armature and a rotating field, or a rotating armature and a stationary field. As modern power system machines (except synchronous converters) have stationary armatures and rotating fields, this arrangement is assumed in the following discussion. The theory determining machine performance is the same in either case.

**Rotor Construction.** There are two general types of rotor construction: (1) the salient- or projecting-pole type, typical of hydro generators, and (2) the cylindrical-rotor (also called round-rotor) type, typical of turbine generators. These two types are shown in Figs. 1 and 2, respectively.

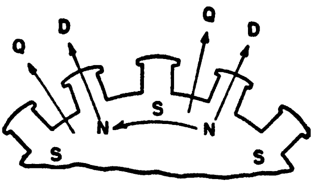

FIG. 1. Cross section of portion of rotor of salient-pole synchronous machine. Direct and quadrature axes indicated by $D$ and $Q$, respectively.

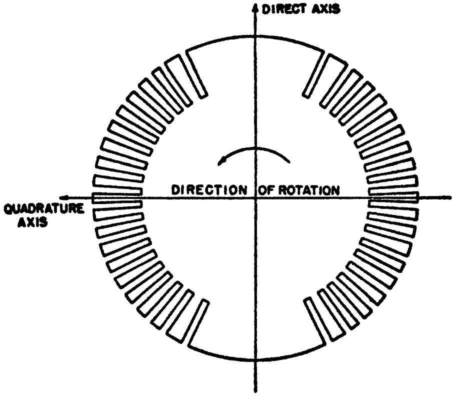

FIG. 2. Cross section of solid rotor of turbo generator showing direct and quadrature axes of rotor.

The rotor of a salient-pole machine consists of a steel spider to which poles of laminated steel, carrying the field winding, are keyed or bolted. The poles are fitted with pole shoes which project beyond the body of the pole and provide a non-uniform air gap, symmetrical about the axis of the pole. They are designed to produce air-gap flux which is approximately cosinusoidally distributed about the axis of the pole.

Rotors of cylindrical-rotor machines are constructed of solid forgings or of thick steel disks suitably bolted together. They are provided with slots at the rotor surface for the field winding. The field winding is not concentrated, as in a salient-pole machine, but is distributed about the axis of the pole. It is held in place in the slots by steel wedges. The spirally wound field produces a field mmf (magnetomotive force) which is approximately cosinusoidally distributed about the direct or pole axis.

**Direct and Quadrature Axes.** Except in special machines not considered here, there are two axes of rotor and air-gap symmetry — the direct axis of the pole and the quadrature axis between poles — and excitation is applied in the direct axis only. The direct and quadrature axes are 90 electrical degrees or $\pi/2$ electrical radians apart, 180 electrical degrees or $\pi$ electrical radians being the distance between adjacent north and south poles. These axes are indicated in Figs. 1 and 2.

**Amortisseur Windings.** Many salient-pole machines have an amortisseur or pole-face winding consisting of non-magnetic bars embedded in the pole face which are connected by end rings to form a short-circuited winding similar to that of a squirrel-cage induction motor. The end rings may or may not form a closed connection between poles. In either case, the pole-face winding, although not uniformly distributed around the periphery of the rotor, is symmetrical about both the direct and quadrature axes of the poles. A salient-pole generator may also have a damping winding consisting of a non-magnetic collar or of short-circuited turns wound about the magnetic pole either above or below the main field winding.

Except in machines especially designed for single-phase loads, amortisseur windings as such are seldom used with cylindrical-rotor machines but, during disturbances, the eddy currents induced in the solid rotor and in the slot wedges over the field windings produce a similar effect. For purposes of visualization or of general analysis, this effect can be represented as that of a pole-face winding.

**Armature Construction.** In a polyphase synchronous machine with the rotor removed, the armature magnetic structure and the armature windings are symmetrical in the phases. The armature core is built up of laminated steel with slots for the armature coils. A coil consists of one or more turns of armature winding. As each slot carries two coil sides, there are the same number of coils as of active slots. An armature winding is distributed if it occupies more than one slot per pole per phase. The number of slots per pole per phase is an integral number in a cylindrical-rotor machine, but in a salient-pole machine it may be a mixed number. In this case, with the rotor rotating at synchronous speed, the armature circuits are not symmetrical in the three phases for each pair of poles but, considering the total winding of each phase, the three phases are symmetrical or so nearly symmetrical as to be so considered during balanced operation or during unsymmetrical faults or other dissymmetries *outside* the machine.

The *pitch* of a coil is the distance between the two coil sides, usually expressed as a fraction of the pole pitch, i.e., distance between adjacent poles. The object of using distributed windings and coils of fractional pitch is to reduce the amplitudes of harmonic armature voltages relative to the amplitude of the fundamental.

The armature winding may be connected with one circuit per phase or with two or more per phase. In a *single-circuit* machine, each phase winding consists of the coils of the phase connected in series. In a *multicircuit* machine, coils are connected in series for each circuit, with the circuits themselves in parallel.

A double-winding machine is a two-circuit machine in which the terminals of the two circuits in each phase are brought out separately. This arrangement enables one three-phase generator to supply two external three-phase circuits which may be independent of each other except for the mutual connection between the two circuits within the generator.

*Single-phase synchronous machines* of both the salient-pole and cylindrical-rotor types are equipped with amortisseur windings. Three-phase cylindrical-rotor machines do not have such windings, and three-phase salient-pole machines may or may not have them. With this distinction, the design of a single-phase machine may be similar to that of a three-phase Y-connected machine with the armature windings of one phase omitted, the generated single-phase voltage being the vector difference between the voltages generated in the two active phases. In calculations involving such single-phase machines in three-phase systems, it may be convenient to treat them as three-phase machines with one phase open.

## Synchronous Speed and Fundamental Frequency

The relation between synchronous speed of the rotor in rpm (revolutions per minute) and fundamental-frequency $f$ in cycles per second, is given by the equation

$$\text{rpm} = \frac{120f}{p} \tag{1}$$

where $p$ is the number of poles.

From [1], at a given frequency, the number of poles is small for a high-speed machine and large for a low-speed one. Cylindrical-rotor machines, which are essentially high-speed machines, are constructed with two or four poles. (In the past, they have had as many as six poles.) For a two-pole 60-cycle machine, the speed is 3600 rpm; at a higher frequency, the speed is proportionally higher. Salient-pole machines are relatively low-speed machines, although there is an overlap in speed between the highest speed of a salient-pole machine and the lowest speed of a cylindrical-rotor machine. Salient-pole machines have been designed with as few as four poles, and as many as 120 poles. A 60-cycle salient-pole generator with 72 poles will have a speed of 100 rpm.

In a power system under normal operation, the rotor of a synchronous machine rotates at synchronous speed, and the average frequency of the system is fundamental frequency. Variations from fundamental frequency may occur but, with the synchronous machines of the system in synchronism, these variations are very small. The average frequency at any system point even over short periods of time, as indicated by the accuracy of electric clocks, is very nearly constant.

## Effect of Operating Conditions upon Machine Characteristics Needed

Many different quantities are required in analyzing the performance of a synchronous machine under all possible conditions of operation, but all of these are not required for a specified operating condition. For example, the characteristics which determine the behavior of a synchronous machine during normal balanced operation are not those used in the study of transient phenomena, nor in determining initial currents and voltages during faults. After a disturbance, some of the induced currents in the rotor circuits may die away very rapidly if there are closed circuits on the rotor in addition to the main field winding. Amortisseur windings and field collars on the salient-pole machine, and the iron rotor itself of the cylindrical-rotor machine, are closed rotor circuits in which the currents induced during disturbances decay much more rapidly than in the main field winding. Machine characteristics which depend upon these rapidly decaying currents are called *subtransient* characteristics; those depending upon components which decay slowly are *transient* characteristics; those depending upon sustained components are *synchronous or steady-state* characteristics.

The most descriptive characteristics of a synchronous machine are the reactances, viewed from the armature terminals, to fundamental-frequency currents during sustained, transient, and subtransient conditions. During disturbances, the time constants which determine the rates of decay of components of current and voltage are also of importance.

A synchronous machine differs from a static network in that the rotor and the rotor circuits move relative to the stator circuits. Because of this motion, the self-inductances of the armature circuits and the mutual inductances between them and between armature and rotor circuits vary with the position of the rotor. They also vary with the degree of saturation in the iron and are influenced by hysteresis. As the synchronous machine has two axes of rotor and air-gap symmetry, the components best suited to its analysis are direct-axis, quadrature-axis, and zero-sequence components. These components will be discussed in connection with various operating conditions, and their relation to symmetrical components pointed out.

Analyses of synchronous machines are greatly simplified if the effects of saturation, hysteresis, and eddy currents in the iron are neglected, these effects to be included later, if of importance. The importance of these neglected effects varies with the problem and the operating conditions. They are relatively unimportant in the calculation of initial symmetrical rms short-circuit currents of fundamental frequency, especially if the fault is separated from the synchronous machine by transformers and transmission lines. On the other hand, it is usually necessary to include the effects of saturation during sustained operating conditions, whether normal or abnormal. An approximate method of including the effects of hysteresis and eddy current losses is to use an effective armature resistance instead of the copper resistance. Approximate methods of including saturation are discussed later in this chapter, and illustrated for steady-state condition in Problem 1.

Before the characteristic constants which apply to various conditions of operation are defined, the base or unit quantities used here will be defined, and a discussion of the normal operation of a three-phase synchronous machine will be given. In this discussion, the harmonics which may be present are pointed out and their relative importance or unimportance noted. For a more complete analysis see reference 1.

**Base or Unit Quantities.** Because of the large number of quantities used in analyzing the performance of synchronous machines, the per unit method of representing them is especially advantageous. Machine constants in per cent or per unit furnished by designers are based on a given rating; these quantities are readily transferred to the base quantities to be used in a system study. (See Volume I, page 19.)

In a three-phase machine, unless otherwise specified,

Unit armature voltage is rated phase voltage.
Unit armature current is rated phase current.
Unit kva per phase is one-third the rated three-phase kva.
Unit power per phase in kilowatts is numerically equal to unit kva per phase.

Unit armature impedance is the ratio of unit armature voltage to unit armature current.

Unit frequency is rated frequency.

Unit speed is rated synchronous speed.

Unit time is the time required for the rotor to pass through one electrical radian when rotating at rated speed. It is also the time required to complete $1/(2\pi)$ cycle of fundamental frequency.

Unit field current is here defined as the field current which produces unit fundamental-frequency armature voltage on open circuit with the rotor rotating at rated speed and saturation neglected.

Unit space fundamental mmf is the resultant fundamental mmf space wave produced by rated three-phase armature currents.

Unit mmf is the maximum value of unit space fundamental mmf wave.

Unit space fundamental flux is the fundamental flux wave which rotating at unit speed induces unit voltage in the armature.

Unit flux linkages with any armature phase are the linkages produced by unit space fundamental flux when its crest lines up with the axis of the phase.

Unit flux density is the maximum density of the unit fundamental flux wave.

Unit permeance coefficient is that permeance coefficient which, multiplied by unit mmf, gives unit flux density.

Time, as defined above, is in electrical radians. When the instantaneous value $i$ of a sinusoidal current of amplitude $I_m$ is written $i = I_m \cos \omega t = I_m \cos 2\pi ft$, $t$ is in seconds, $f$ is cycles per second, $2\pi$ is radians per cycle, and $\omega t$ is in radians. It is more convenient, especially if differentiation or integration is involved, to replace $\omega t$ by a new variable $t$, where $t$ is time in electrical radians.

Unit field current, as defined above, is useful in many system studies. It should be pointed out, however, that in some studies a different value of unit field current may be found more convenient. The use of a different unit field current will be illustrated later.

## Normal Operation of a Three-Phase Synchronous Machine

Under normal operation in a symmetrical three-phase system, the rotor of a synchronous machine rotates at synchronous speed, the field winding is excited, and the armature currents are substantially fundamental-frequency currents of positive-sequence phase order. The mmf's, flux linkages, and armature voltages resulting from field and armature currents will be considered separately; then, under the assumption of negligible saturation and no hysteresis, superposition will be applied to determine the resultant armature terminal voltage.

**Field Current Phenomena.** With the armature on open circuit, the field current is a d-c current produced by the field exciter. By varying the exciter voltage, any desired field current up to maximum field can be obtained. The field mmf is determined by the field ampere-turns. Field flux distribution can be determined graphically by means of flux plots.[2]

Flux linkages with the field, resulting from field mmf, consist of air-gap fluxes and leakage fluxes in the interpolar spaces and ends of the machine. Field flux which crosses the air gap into the armature and returns by way of the air gap to adjacent poles, links both armature and field. This air-gap flux mutual to armature and field is approximately, but not strictly, cosinusoidally distributed about the direct axes of the poles. It can be resolved by harmonic analysis into a fundamental space wave of flux and odd-harmonic space waves, all stationary with respect to the rotor and cosinusoidal in their distribution about the direct axis. Of these harmonic space waves, the third is usually the most pronounced.

With the rotor rotating at synchronous speed in its normal direction of rotation and the armature on open circuit, the fundamental space wave of flux in the air gap generates fundamental-frequency armature voltages in the three phases which are called the excitation or generated voltages of the machine. The phase order of these generated voltages determines positive-sequence phase order.

The harmonic space waves of flux, traveling at synchronous speed with the rotor, generate corresponding time-harmonic voltages in the armature. Harmonic armature voltages are reduced relative to the fundamental by the use of armature coils of fractional pitch and a distributed winding. In general they are small and, except for the third harmonic, need not be considered unless there is resonance or near resonance to some particular harmonic.

In a three-phase machine, the time in radians required for a pole of the rotor rotating at synchronous speed to pass from the axis of one phase to that of the next phase is $\frac{2}{3}\pi$ electrical radians. As the period of third harmonic armature voltages is also $\frac{2}{3}\pi$ radians, the third harmonic armature voltages generated in the three phases by the third harmonic space wave of flux, traveling at synchronous speed with the rotor, will be in phase. Being equal in magnitude and in phase, they are *zero-sequence third harmonic voltages.* (The fifth harmonic voltages are of negative-sequence phase order, the seventh of positive-sequence phase order, and the ninth of zero-sequence phase order, etc.)

If the machine is Δ-connected, third harmonic zero-sequence voltages are consumed by third harmonic $IZ$ drop in the Δ and do not appear at the machine terminals or in the connected circuits. If the neutral of a Y-connected machine is grounded, and there is a path for third harmonic zero-sequence currents in the connected system, third harmonic zero-sequence currents will flow of magnitude directly proportional to the generated third harmonic voltage and inversely proportional to the third harmonic zero-sequence impedance of the total path. If there is no path for zero-sequence current (as when the grounded generator supplies a transformer Δ-connected on the generator side), zero-sequence third harmonic voltages will exist at the generator terminal; but like fundamental-frequency zero-sequence voltages, they will not be present in the line-to-line voltages. If the Y-connected machine is ungrounded, a zero-sequence third harmonic voltage will exist at its neutral unless made negligible by pitch and distribution factors.

**Armature Current Phenomenon.** The armature mmf's of all coils may be superposed and referred to the armature surface if it is assumed that the armature reluctance is negligible and there are no stator slots, the armature current in each slot being concentrated in the center line of the slot at the armature surface. In plotting space waves of armature mmf, the ordinate will indicate the strength of mmf and the abscissa the angular distance along the armature surface in electrical radians from a chosen point of reference.

The space distribution of the mmf of a single coil is represented by a rectangular space wave between coil sides, with angular distance along the armature surface as abscissa and the mmf of the coil as ordinate. With sinusoidal current in the coil, its mmf space wave pulsates sinusoidally in time with the current. With only positive-sequence currents of fundamental frequency in the armature, superposition of the mmf space waves of all coils of each of the three phases results in three identical pulsating mmf space waves, displaced from each other by $\frac{2}{3}\pi$ electrical radians and pulsating in time with their currents, which are $\frac{2}{3}\pi$ electrical radians apart in time phase. The three stationary-in-space mmf waves are made approximately sinusoidal by the use of armature coils of fractional pitch and a distributed winding. With a given coil pitch, the waves become more nearly sinusoidal as the number of slots per pole per phase is increased. As they are not strictly sinusoidal, each of the three mmf space waves may be resolved by harmonic analysis into a fundamental space wave and odd-harmonic space waves, cosinusodially distributed about the axis of magnetization of their respective phases and pulsating in time with the currents in these phases.

Proof is given in Appendix B of the following statements:

1. The fundamental (first harmonic) and each of the other odd-harmonic stationary-in-space pulsating mmf waves can be replaced by two component waves of one-half amplitude, one rotating forward and one backward relative to the armature. With phase order *abc*, a forward rotating wave passes the axes of the phases in the order *abc*. This is also the normal direction of rotation of the rotor. See Fig. 3.

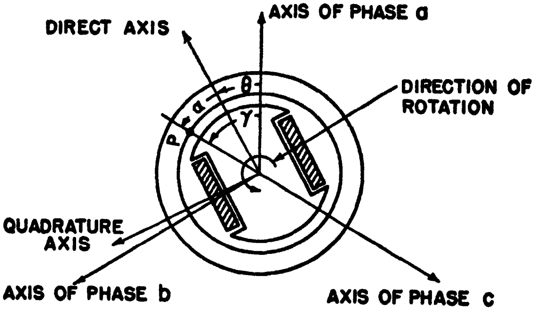

Fig. 3. Equivalent two-pole salient-pole synchronous machine.

2. Superposition of the forward- and backward-rotating harmonic mmf space waves of the three phases results in the elimination of some of them. The forward-rotating fundamental mmf waves in the three phases add to give a resultant fundamental mmf wave which rotates forward at synchronous speed relative to the stator. Its amplitude is $\frac{3}{2}$ the amplitude of the pulsating, stationary-in-space fundamental mmf wave of any phase. The backward-rotating fundamental mmf space waves of the three phases add to zero. Both the forward- and backward-rotating third-harmonic space waves, and all multiples of the third, vanish. Relative to the stator, the resultant fifth harmonic space wave rotates backward at one-fifth synchronous speed, the seventh forward at one-seventh synchronous speed, the eleventh backward at one-eleventh synchronous speed, the thirteenth forward at one-thirteenth synchronous speed, etc.

Let $\gamma$ = the angular displacement in electrical radians of any point $P$ on the armature surface from the axis of phase $a$, measured in the forward direction. (See Fig. 3.)

$t$ = time in electrical radians from the instant the current in phase $a$ reaches its crest value.

The instantaneous mmf $a$ at $P$ is shown in Appendix B to be

$$a = A_1 \cos(\gamma - t) + A_5 \cos(5\gamma + t) + A_7 \cos(7\gamma - t) + A_{11} \cos(11\gamma + t) + A_{13} \cos(13\gamma - t) + \cdots \quad [2]$$

where $A$ is per unit amplitude of mmf space waves, and numerical subscripts indicate the harmonic order. The space waves of mmf given by [2] are independent of the position and motion of the rotor.

Equation [2] may be rewritten

$$a = A_1 \left[ \cos(\gamma - t) + \frac{C_5}{5} \cos(5\gamma + t) + \frac{C_7}{7} \cos(7\gamma - t) + \frac{C_{11}}{11} \cos(11\gamma + t) + \cdots \right] \quad [2a]$$

where $C_n = \dfrac{(kp)_n (kd)_n}{(kp)_1 (kd)_1}$

$n$ = order of harmonic

$(kp)_n$ = pitch factor for harmonic indicated by subscript

$$= \frac{\sin(n \times \text{per unit pitch} \times 90^\circ)}{\sin(n \times 90^\circ)}$$

$(kp)_1$ = pitch factor for fundamental = $(kp)_n$ when $n = 1$

$(kd)_n$ = distribution factor for harmonic indicated by subscript

$$= \frac{\sin \dfrac{n \times 90^\circ}{\phi}}{S \sin \dfrac{n \times 90^\circ}{S\phi}}$$

$(kd)_1$ = distribution factor for fundamental = $(kd)_n$ when $n = 1$

$\phi$ = number of phases

$S$ = number of coils per pole per phase.

From [2] and [2a],

$$A_5 = \frac{C_5}{5} A_1; \quad A_7 = \frac{C_7}{7} A_1; \text{ etc.}$$

With a full-pitch, non-distributed winding, all pitch and distribution factors are unity, and the amplitude $A_n$ of any harmonic mmf space wave in [2] is $A_1/n$. In a machine of fractional pitch and a distributed winding, $A_n$ is further reduced by pitch and distribution factors. In a machine with a sinusoidally distributed armature winding, all terms in [2] after the first will be zero. In the usual machine, the amplitude of these mmf waves are small relative to the fundamental.

With all quantities in per unit as previously defined, $A_1$, the per unit crest of the fundamental mmf space wave, is equal to per unit positive-sequence current. As the rms value of a fundamental-frequency current in per unit of the rms value of base current is the same as its crest in per unit of the crest of base current, it is unnecessary to specify whether the per unit value of crest or rms current is under consideration.

With the rotor rotating at synchronous speed in its normal direction of rotation, the fundamental mmf space wave $A_1 \cos (\gamma - t)$, which rotates forward at synchronous speed with respect to the armature, is stationary with respect to the rotor. Its crest may have any angular displacement from the direct-pole axis.

Let $\theta$ = angular displacement in electrical radians at any time $t$ of the direct axis from the axis of phase $a$, measured in the normal direction of rotation of the rotor. (See Fig. 3.)

At time $t = 0$, $\theta = \theta_0$. At any subsequent time,

$$\theta = \theta_0 + t \qquad [3]$$

With the crest of the fundamental space wave of mmf at the axis of phase $a$ at time $t = 0$, its displacement from the direct axis of the pole is $-\theta_0$. In Fig. 4($a$), $\theta_0 = -\pi/6$ radians $= -30°$, and the crest of the mmf wave is 30° ahead of the direct axis. It retains this position as it rotates in synchronism with the rotor.

## Direct- and Quadrature-Axis Components

The rotor magnetic structure and the permeance (neglecting stator slots) are symmetrical about both the direct and the quadrature axes of the poles; therefore, to obtain the simplest expressions for the flux distribution in the air gap, the rotating mmf space waves can be referred to the rotor, and the permeance coefficients which correspond to points along the armature surface at definite angular displacements from the direct axis of the pole can be applied.

Blondel[3] treated salient-pole machines by resolving the fundamental space component of armature mmf $A_1 \cos (\gamma - t)$ into two fundamental space components, one with its crest $A_{1d}$ in line with the direct axis of the pole and the other with its crest $A_{1q}$ in line with the quadrature axis of the pole. The two space components of the fundamental space wave of the armature mmf of Fig. 4(*a*) are shown in Figs. 4(*b*) and 4(*c*), where $A_{1d} = A_1 \cos \theta_0$, and $A_{1q} = -A_1 \sin \theta_0$, $\theta_0 = -30°$, and the quadrature axis is taken *ahead* of the direct axis in the normal direction of rotation of the rotor. Blondel's method of analysis was later extended by Doherty and Nickle,[1] who began their analysis by resolving the armature current into direct-axis and quadrature-axis components of current.

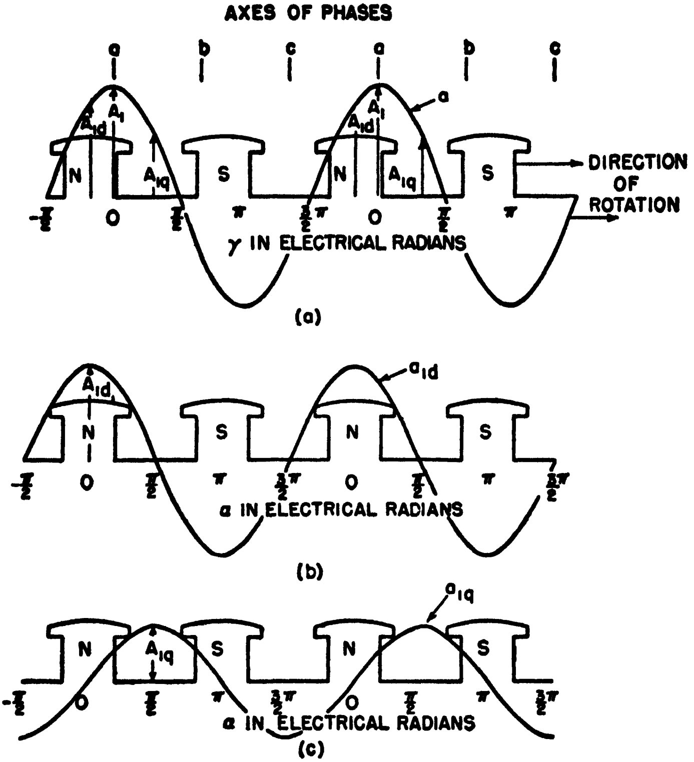

FIG. 4. (*a*) Position of fundamental mmf space wave and axes of rotor relative to axes of phases at time $t = 0$ when current in phase $a$ is maximum and $\theta_0 = -\pi/6$ radians. (*b*) and (*c*) Direct- and quadrature-axis components, respectively, of fundamental mmf space wave of (*a*).

With the rotor rotating at synchronous speed, positive-sequence currents of rated frequency in the three phases produce a fundamental space wave of mmf which is fixed in space relative to the rotor. See Fig. 4(*a*). The direct- and quadrature-axis space components of this fundamental mmf wave are likewise fixed relative to the rotor. See Figs. 4(*b*) and 4(*c*). As these component mmf waves do not pulsate, they correspond to waves produced by constant currents with axes of magnetization which line up with the direct and quadrature axes of the poles and rotate in synchronism with the rotor.

**Direct- and Quadrature-Axis Components of Armature Current.** These are components of current which magnetize in the direct and quadrature axes, respectively, of the rotor. At any instant, they produce the same mmf space waves in the direct and quadrature axes of the rotor as are produced by the actual a-c currents in the three phases. With *sustained positive-sequence currents of rated frequency* in the armature and the currents of all phases considered together, direct-axis armature current is a constant current equal to the per unit value of current in any phase at the instant the direct axis of the pole lines up with the axis of that phase; quadrature-axis armature current is a constant current equal to the per unit value of the current in any phase at the instant the quadrature axis lines up with the axis of that phase. In the general case, there will be both direct- and quadrature-axis components of armature current.

Direct- and quadrature-axis components of current are indicated by $i_d$ and $i_q$, respectively; $i_d$ is positive if it magnetizes in a direction *opposite* to that of the main field winding; $i_q$ is positive if it magnetizes ahead of positive $i_d$ in the normal direction of rotation of the rotor. This choice for positive $i_d$ and positive $i_q$ is arbitrary. Figure 5 shows the direction of magnetization of positive $i_d$ and positive $i_q$ relative to that of the main field winding. Figure 5 is consistent with Fig. 3, in which the quadrature axis is shown ahead of the direct axis in the normal direction of rotation of the rotor; it is also consistent with Figs. 4(*b*) and 4(*c*) in which $A_{1d} = i_d$ and $A_{1q} = i_q$ are shown positive when magnetization is opposite to that of the main field winding. In an overexcited machine, $i_d$ is positive; in an underexcited machine, $i_d$ is negative. When the quadrature axis is assumed ahead of the direct axis, $i_q$ is positive in a machine supplying positive power and negative in a machine supplying negative power (i.e., a motor receiving positive power). With all quantities in per unit of their respective base or unit values, as previously defined, one vector in a per unit vector diagram may represent more than one quantity. Thus in Fig. 5, $I_f$ represents per unit field current, the per unit crest of the fundamental space wave of flux in the air gap produced by field current and the maximum value of the per unit field flux linkages with any armature phase; $i$ represents per unit rms positive-sequence armature current and the per unit crest $A_1$ of the fundamental space wave of mmf produced by this current; $i_d$ and $i_q$ represent per unit components of the rms current $i$, per unit components of the armature current of the three phases which magnetize in the direct and quadrature axes, respectively, and also the per unit crest $A_{1d}$ and $A_{1q}$ of the fundamental space waves of mmf produced by positive-sequence currents in these axes. This discussion of per unit vector diagrams is continued in connection with Fig. 10(*a*) where per unit voltages are added to the vector diagram of Fig. 5.

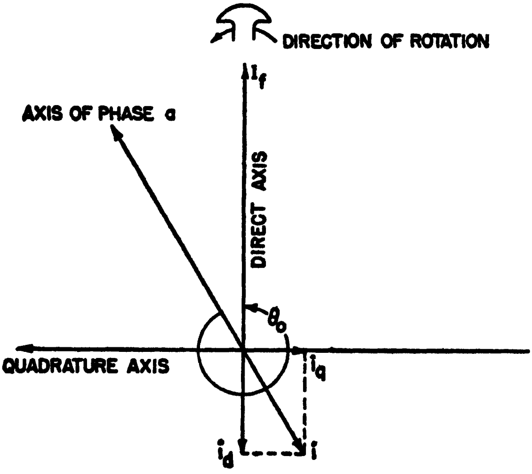

FIG. 5. Per unit diagram showing direction of magnetization of sustained positive-sequence armature current $i$ and its components $i_d$ and $i_q$, both positive, relative to the direction of magnetization of d-c field current $I_f$.

**Direct- and Quadrature-Axis Components of Mmf.** The direct-axis armature current $i_d$ produces a fundamental space wave of mmf, stationary with respect to the rotor, which has its crest $A_{1d}$ in line with the direct axis of the poles. See Fig. 4(*b*). It also produces a series of rotating odd-order space harmonics of mmf — a fifth rotating backward at one-fifth synchronous speed relative to the stator, a seventh rotating forward at one-seventh synchronous speed, etc. Similarly, the quadrature axis armature current $i_q$ produces a fundamental space wave of mmf, stationary with respect to the rotor, which has its crest $A_{1q}$ in line with the quadrature axis of the poles, Fig. 4(*c*), and also a series of rotating odd-order space harmonics of mmf. By resolving the armature current into direct- and quadrature-axis components, the fundamental space wave of mmf and also the harmonic space wave of mmf are resolved into direct- and quadrature-axis space components.

When positive-sequence currents are replaced by their direct-axis and quadrature-axis components of current, $i_d$ and $i_q$, respectively, the instantaneous mmf's $a_d$ and $a_q$ at any point $P$ on the armature surface produced by $i_d$ and $i_q$, respectively, can be written by analogy from [2]. The following equations give $a_d$ and $a_q$ referred to the armature, where time $t$ is reckoned from the instant the direct axis lines up with the axis of phase $a$ and the current in phase $a$ is then entirely direct-axis current, and the quadrature axis is taken ahead of the direct axis in the normal direction of rotation of the rotor.

$$a_d = A_{1d} \cos (\gamma - t) + A_{5d} \cos (5\gamma + t) + A_{7d} \cos (7\gamma - t) + \cdots \quad [4]$$

$$a_q = -A_{1q} \sin (\gamma - t) - A_{5q} \sin (5\gamma + t) - A_{7q} \cos (7\gamma - t) - \cdots \quad [5]$$

In [4] and [5], $a_d$ and $a_q$ are referred to the armature. Let

$\alpha$ = the angular displacement of point $P$ from the direct axis of the rotor in the positive direction (see Fig. 3)

By definition, and from [3],

$$\gamma = \alpha + \theta = \alpha + \theta_0 + t \quad [6]$$

With time reckoned from the instant the direct axis lines up with the axis of phase $a$, $\theta_0 = 0$ and $\gamma = \alpha + t$. If $\gamma$ in [4] and [5] is replaced by $\alpha + t$, $a_d$ and $a_q$ have the following values:

$$\begin{aligned} a_d = {} & A_{1d} \cos \alpha + A_{5d} \cos (5\alpha + 6t) + A_{7d} \cos (7\alpha + 6t) \\ & + A_{11d} \cos (11\alpha + 12t) + \cdots \end{aligned} \quad [7]$$

$$\begin{aligned} a_q = {} & -A_{1q} \sin \alpha - A_{5q} \sin (5\alpha + 6t) - A_{7q} \sin (7\alpha + 6t) \\ & - A_{11q} \sin (11\alpha + 12t) + \cdots \end{aligned} \quad [8]$$

Equations [7] and [8] give the instantaneous mmf at any point $P$ on the armature surface displaced from the direct-axis of the rotor by the angle $\alpha$. The fundamental mmf space waves $A_{1d} \cos \alpha$ and $- A_{1q} \sin \alpha$ are stationary with respect to the rotor and therefore meet constant permeances. When expressed in per unit, $A_{1d} = i_d$ and $A_{1q} = i_q$. The harmonic mmf space waves rotate with respect to the rotor but, by trigonometric transformations,* each can be resolved into two pulsating waves stationary with respect to the rotor so that they also meet constant permeances. The following equations for $a_d$ and $a_q$ then replace [7] and [8]:

$$\begin{aligned} a_d = {} & A_{1d} \cos \alpha + A_{5d} (\cos 5\alpha \cos 6t - \sin 5\alpha \sin 6t) \\ & + A_{7d} (\cos 7\alpha \cos 6t - \sin 7\alpha \sin 6t) + \cdots \end{aligned} \quad [9]$$

$$\begin{aligned} a_q = {} & -[A_{1q} \sin \alpha + A_{5q} (\sin 5\alpha \cos 6t + \cos 5\alpha \sin 6t) \\ & + A_{7q} (\sin 7\alpha \cos 6t + \cos 7\alpha \sin 6t) + \cdots] \end{aligned} \quad [10]$$

---

* $\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b$

For each rotating space harmonic wave of mmf in [7] and [8] there are two stationary (with respect to the rotor) mmf space waves in [9] and [10] which pulsate in time at 6, or multiples of 6, times fundamental frequency.

It is of interest at this point to note that these harmonic mmf space waves, stationary but pulsating with respect to the poles, produce flux waves also stationary but pulsating with respect to the poles which induce sixth and multiples of the sixth time-harmonic voltages in the field which would give rise to corresponding time harmonics in the field current if they were appreciable. In general, these induced field currents are negligible because of the low amplitudes of the mmf waves and the high impedance of the field winding to high harmonic currents; however, a sixth harmonic ripple in the field current has been noted[1] in rare cases during normal operation. The fundamental mmf waves, being stationary with respect to the poles, produce flux linkages which are also stationary with respect to the poles. No voltage is therefore induced in the field winding by them. The field current under normal balanced operation may be considered a constant d-c current.

**Permeance.** The mmf at any point along the armature surface multiplied by the permeance applying to the magnetic path at that point gives the flux density at the point. Permeance varies from point to point but, neglecting stator slots, is symmetrical about both the direct and the quadrature axes of the poles; permeance can therefore be represented by a Fourier series of even cosines. The permeance coefficients in the permeance equation are different for each *space harmonic* of mmf (fundamental included), but it has been shown[1] that they are almost exactly the same in a given machine for cosine and sine distribution of mmf about the direct axis of the pole for a given space harmonic of mmf (except for the fundamental). Methods of determining permeance coefficients by means of flux plots are given in reference 1.

Permeance $p$ at any point $P$ on the armature surface is given by a series of the form

$$(p)_n = (p_0)_n + (p_2)_n \cos 2\alpha + (p_4)_n \cos 4\alpha + (p_6)_n \cos 6\alpha + \cdots \quad [11]$$

where $n$ is the order of the space harmonic of mmf; $\alpha$ is the electrical angle between the direct axis of the pole and any point $P$ on the armature surface, measured in the positive direction of rotation of the rotor (see Fig. 3); and $p_0, p_2, p_4, \cdots$ are permeance coefficients.

In determining fundamental-frequency armature voltages, only the first two terms of [11] are required. If the terms in [11] after the second are omitted, the permeance equation for the *fundamental space harmonic of mmf*, with subscript $n = 1$ omitted, is

$$p = p_0 + p_2 \cos 2\alpha \qquad [12]$$

When, as here, the effects of d-c field and a-c armature currents are considered separately, the coefficients $p_0$ and $p_2$ for the sustained fundamental space waves of mmf are not the same as for the fundamental mmf space waves which appear suddenly and give rise to currents in the rotor circuits, thereby effectively modifying the permeance; nor are they the same as under transient conditions.

**Armature Flux Linkages and Reactive Armature Voltages.** When each fundamental and odd-harmonic mmf space wave in [9] and [10] is multiplied by its corresponding permeance function in the form of [11], the flux distribution in the air gap referred to the rotor is obtained. In addition to this flux, there is the armature leakage flux in the slots and ends of the machine.

The fundamental mmf space waves, $a_{1d} = A_{1d} \cos \alpha$ and $a_{1q} = -A_{1q} \sin \alpha$, when multiplied by their permeance function in the form of [11], will produce fundamental, third, fifth, etc., space waves of flux in the direct and quadrature axes. The flux waves produced by the fundamental space waves of armature mmf are the only armature flux waves constant in time with respect to the poles and therefore the only ones that cause a constant value of mutual linkages between armature and rotor circuits. Rotating with the rotor at synchronous speed, the fundamental space waves of flux produce fundamental-frequency reactive armature voltages. The corresponding reactances are the reactances $x_{ad}$ and $x_{aq}$ in the direct and quadrature axes, respectively. The harmonic space waves of flux, resulting from the fundamental space waves of mmf $a_{1d}$ and $a_{1q}$ and rotating with the rotor, produce time-harmonic armature voltages of the same harmonic order which are reduced in magnitude relative to the fundamental by the order of the harmonic and by pitch and distribution factors. These harmonic armature voltages are of the same nature as those produced by harmonic space waves of field flux, discussed under "Field Current Phenomena"; their magnitudes, however, depend upon the magnitude of the armature current, whereas the magnitudes of the voltages produced by field flux depend upon the magnitude of the field current.

Each pair of pulsating harmonic mmf space waves given in [9] and [10], when multiplied by their permeance function in the form of [11], produces pairs of fundamental, third, fifth, etc., space waves of flux all of which are pulsating with respect to the rotor. These pulsating flux waves can be resolved into rotating sinusoids of flux, some rotating forward with respect to the poles and others backward. The rotating flux waves can be referred to the armature, and expressed as pulsating flux waves from which the induced armature voltages can be obtained. (See reference 1 for detailed treatment.) Of these many rotating sinusoids of flux, a few produce *fundamental-frequency armature voltages;* the others produce time-harmonic voltages in the armature.

The *fundamental-frequency armature voltages* resulting from the rotating harmonic fluxes in the air gap are of small magnitude. Since these rotating air-gap fluxes do not cause an average or constant value of mutual linkages with the rotor circuits, the corresponding fundamental-frequency armature voltage is defined as the reactive voltage of *differential air-gap leakage reactance.* This air-gap leakage reactance added to the slot-leakage and end-leakage terms gives total armature leakage reactance $x_{\ell}$. The reactive armature voltage corresponding to $x_{\ell}$ is the reactive voltage of armature leakage reactance.

The *harmonic armature voltages* resulting from fundamental-frequency positive-sequence armature currents, whether produced by harmonic space waves of flux which rotate with the rotor or by the rotating sinusoids of flux in the air gap, are usually of small magnitude in modern polyphase machines, their amplitudes relative to the fundamental being reduced by the order of the harmonic and pitch and distribution factors.

The armature terminal voltage is determined by the time rate of change of the net air-gap flux linking the armature and the armature leakage flux. The net air-gap flux is due to currents in both field and armature. Instead of determining the voltage resulting for the net air-gap flux, the procedure outlined here is to determine the vector sum of the voltages resulting for d-c field current and armature currents considered separately.

**Summary and Conclusions.** The air-gap flux produced by the d-c field consists of a fundamental space wave of flux plus odd-order harmonic space waves of flux. The fundamental generates fundamental-frequency armature voltage; the harmonic space waves of flux generate corresponding time harmonics which are usually of small magnitude relative to the fundamental. The armature voltage generated by the fundamental wave of field flux is called the *generated or excitation voltage* of the machine. The phase order of this voltage determines positive-sequence phase order.

Positive-sequence armature currents of fundamental frequency produce a fundamental space wave of mmf which rotates forward at synchronous speed relative to armature and therefore rotates in synchronism with the rotor. This is the armature reaction mmf. There are also rotating space harmonics of mmf due to non-sinusoidal distribution of the armature winding. If positive-sequence currents are replaced by their direct- and quadrature-axis components of current, these components of current produce fundamental mmf space waves which rotate with their crests in line with the direct axis of the pole and the quadrature axis between poles, respectively. These fundamental mmf space waves are referred to as the direct- and quadrature-axis components of armature reaction mmf.

The fundamental space component of air-gap flux resulting from the direct-axis armature reaction mmf is the only armature flux which produces both a fundamental-frequency armature voltage and a constant value of flux linkages in the main field winding. The corresponding reactance is the reactance $x_{ad}$. In the quadrature axis, there is likewise a fundamental space component of air-gap flux which is mutual to armature and rotor in the quadrature axis; the corresponding reactance is the reactance $x_{aq}$. By adding the leakage reactance $x_\ell$ to $x_{ad}$ and $x_{aq}$, the reactances $x_d$ and $x_q$ to sustained direct- and quadrature-axis components of current, respectively, are obtained.

The harmonic armature voltages of a synchronous machine under normal operation in a symmetrical three-phase system are usually of small magnitude relative to the fundamental; the armature currents and voltages are therefore substantially sinusoidal. In calculating the fundamental-frequency terminal voltage of a synchronous machine under balanced conditions, it is usually necessary to consider only the excitation voltage generated by the fundamental flux wave of the field, and the reactive and resistive voltages of fundamental frequency produced by direct- and quadrature-axis components of armature current. This is illustrated in Fig. 10(*a*) and will be fully discussed after the reactances and resistances of synchronous machines have been defined.

## CHARACTERISTICS OF SYNCHRONOUS MACHINES

**No-Load Saturation Curve and Air-Gap Line.** In Fig. 6, the curve *OSM* is the no-load saturation curve in which field current is abscissa and rms armature voltage at rated frequency on open circuit is ordinate. The *air-gap line OAN* is the extended straight-line part of the no-load saturation curve. The general shape of no-load saturation curves is the same in all synchronous machines, but the slope of the air-gap line and the rate at which the no-load saturation curve bends over will differ in machines of different design.

**Synchronous Impedance Curve.** In Fig. 6, $OB$ is the synchronous impedance curve in which armature current at rated frequency on sustained three-phase short-circuit at the machine terminals is ordinate and field current is abscissa. As there is negligible saturation under a sustained three-phase short circuit in the usual machine, the synchronous impedance curve is a straight line; it may, however, begin to bend over slightly in some machines at high field currents. Synchronous impedance in ohms is the ratio of the phase voltage on the air-gap line in volts to the phase current in amperes on the synchronous impedance curve (straight-line part) determined at the same field current. Synchronous impedance is the same at all field currents as long as the synchronous impedance curve is a straight line, indicating no saturation.

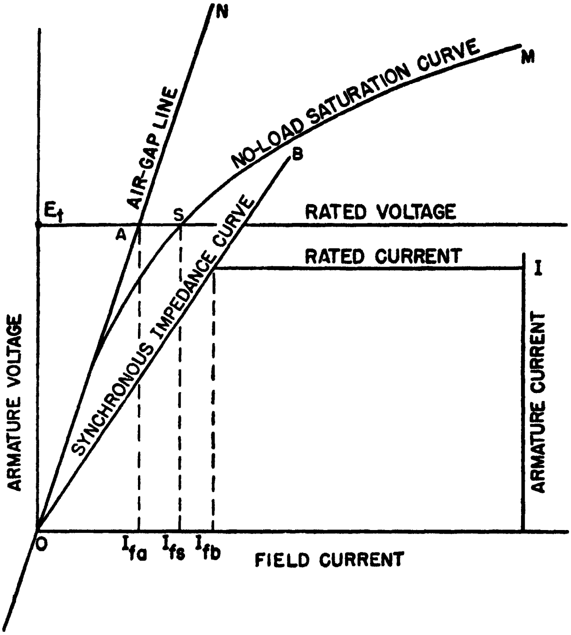

FIG. 6. No-load saturation curve of a synchronous machine at rated frequency, with air-gap line and synchronous impedance curve indicated.

It is convenient to plot the no-load saturation and synchronous impedance curves in per unit. In that case, armature volts and amperes in Fig. 6 will be expressed in per unit of rated phase voltage and current, respectively, and field amperes in per unit of *that field current which gives rated armature voltage on open circuit as determined from the air-gap line.* In Fig. 6, $E_t$, $I$, and $I_{fa}$ will then be unity.

**Per Unit Synchronous Impedance** is usually defined as the ratio of the field current at rated armature current on a sustained three-phase short circuit at the machine terminals to the field current required to produce rated open-circuit voltage on the air-gap line. From Fig. 6,

$$\text{Per unit synchronous impedance} = \frac{I_{fb}}{I_{fa}}$$

**Short-Circuit Ratio** (SCR) is the ratio of the field current corresponding to rated armature voltage on open circuit read from the *no-load saturation curve* to the field current required to produce rated armature current on sustained three-phase short circuit at the machine terminals, both determined at rated frequency. From Fig. 6,

$$\text{SCR} = \frac{I_{fs}}{I_{fb}}$$

The short-circuit ratio includes some saturation. If there were no saturation, the *reciprocal of the short-circuit ratio* 1/(SCR) would equal the per unit synchronous impedance.

**Potier Reactance** $x_p$ is a machine reactance which may be obtained from two test curves. One of these curves is the no-load saturation curve already described. The other is the zero-power-factor (over-excited) saturation curve at rated current. These two curves are shown in Fig. 7. For both curves, terminal voltage in per unit of rated voltage is ordinate, and field current in per unit of the field current required to produce rated voltage on the air-gap line is abscissa. The Potier reactance is determined as follows.

From Point $B$, at rated terminal voltage $V_t$ on the zero-power-factor curve, lay off horizontally a distance $BA$ equal to $OS$, the per unit field current $I_{fz}$ required to force rated current through synchronous impedance at zero terminal voltage. Through $A$, draw $AC$ parallel to the air-gap line and intersecting the no-load saturation curve at $C$. From $C$ drop a perpendicular $CD$ to the horizontal line $AB$, intersecting it at $D$. The triangle $ABC$ is called the Potier triangle. $CD$, the height of the Potier triangle, is the Potier voltage. $CD$ divided by rated armature current is the Potier reactance. As per unit quantities are used in Fig. 7 and rated armature current is unity, $CD$ is the Potier reactance in per unit of base armature impedance per phase.

Potier reactance is used in Problem 1 to determine field current corresponding to a given load on the machine at a given terminal voltage with the effect of saturation included.

Unless otherwise specified, Potier reactance is determined from the no-load saturation curve, and the field currents corresponding to rated terminal voltage and rated current at zero-power-factor (overexcited), as explained above. It could, however, be determined in a similar manner at other than rated current and at power factors different from zero.

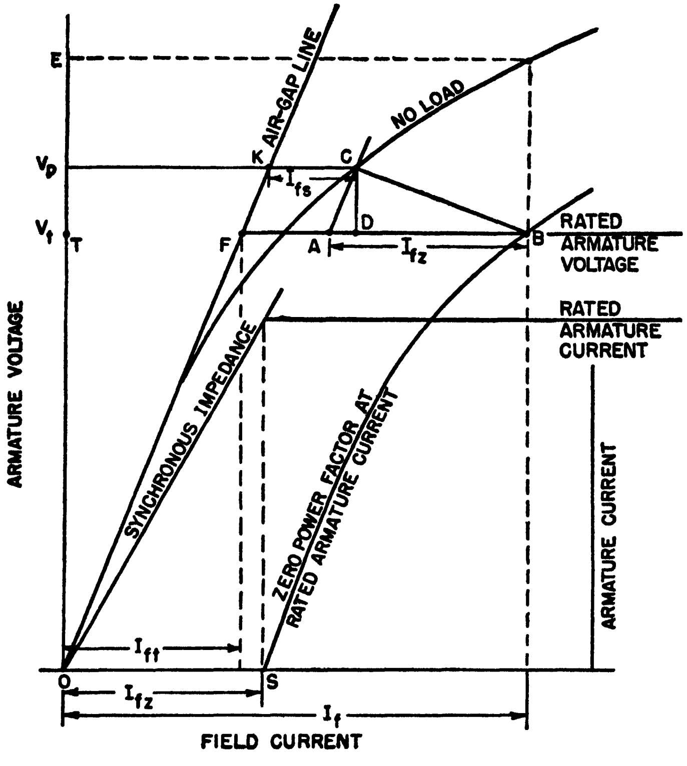

Fig. 7. Curves for determining Potier reactance.

**Armature Leakage Reactance** $x_\ell$ is the reactance corresponding to the fundamental-frequency armature voltage produced by flux which links the armature but does not link the rotor or does not produce constant linkages in it. The differential air-gap leakage reactance added to the slot and end-leakage reactances gives total armature leakage reactance $x_\ell$. Armature leakage reactance is very nearly equal in the two axes; with saturation neglected this reactance is approximately constant in a given machine for transient and subtransient as well as sustained operation, regardless of rotor motion. Positive- and negative-sequence reactances viewed from the armature terminals will include armature leakage reactance in addition to other terms. Leakage reactance cannot be measured directly, but methods for calculating it are available.[4]

## Sequence Reactance of Synchronous Machines at Rated Frequency[5-7]

Reactances, in general, are defined as the ratios of specified voltages or components of voltage to specified currents or components of current. The rated frequency positive-, negative-, or zero-sequence reactances of a synchronous machine will be defined here as the ratio of a specified fundamental-frequency component of reactive armature voltage, due to a specified *applied* fundamental-frequency component of armature current, to this component of current with the rotor rotating at synchronous speed in its normal direction of rotation and the field winding short-circuited through its exciter. Currents may be applied through reactances which offer high impedances to harmonic currents, so that harmonic armature voltages which result from applied currents are not short-circuited and their effects can be considered separately. When a reactance is defined as the ratio of a specified applied component of armature voltage to a specified resultant component of armature current, harmonic voltages which result from the flow of currents are short-circuited by the applied voltages, and the currents due to them may affect the value of the specified component of current. Both ways of defining reactance are found in the literature.[5] In cases where harmonics are unimportant, reactances are substantially the same by either definition, provided they are determined at the same degree of saturation.

**Positive-Sequence Reactances.** If positive-sequence currents of rated frequency are applied to the armature, a fundamental space wave of mmf is produced which rotates forward in synchronism with the rotor. The air-gap permeance offered to this mmf wave depends upon its position relative to the poles and whether the armature currents are suddenly applied or are sustained, and, if suddenly applied, whether the machine has an amortisseur winding or its equivalent.

**Direct-Axis Synchronous Reactance $x_d$** is the ratio of the fundamental-frequency component of reactive armature voltage, due to the fundamental-frequency direct-axis positive-sequence component of armature current, to this component of current under sustained balanced conditions.

The following test can be used to determine $x_d$: With the rotor rotated mechanically at synchronous speed and the field winding unexcited, let sustained direct-axis positive-sequence currents of rated frequency be applied at the machine terminals so that the fundamental space wave of mmf lines up with the direct axis, as shown in Fig. 4(*b*). If the currents are applied from an external source through balanced reactances which offer high impedance to harmonic currents, there will be no appreciable harmonic currents. Any harmonic voltages which result from the flow of the applied currents, however, will be present. Under this condition of operation, the ratio of the resultant fundamental-frequency component of reactive armature voltage in any phase to the current in that phase is the direct-axis synchronous reactance $x_d$.

The fundamental flux wave, rotating with the rotor, is stationary in the direct axis and therefore causes no change in the flux leakages in the field winding. In determining $x_d$ by this test, it is unimportant whether the field winding is open or short-circuited, provided it is unexcited.

When a sustained three-phase short circuit is placed at the terminals of an unloaded synchronous machine rotating at rated speed with the field excited, because of the low ratio of resistance to reactance, the fundamental mmf space wave lines up almost exactly with the direct axis of the pole, indicating direct-axis current. As the harmonic armature voltages resulting from sustained positive-sequence currents are of small magnitude, short-circuiting them does appreciably affect the short-circuit current which, in the usual machine, is substantially positive-sequence current of rated frequency. Direct-axis synchronous reactance by this test is the ratio of the no-load voltage on the air-gap line to the wattless component of short-circuit current. It is the same as the reactance component of synchronous impedance previously defined. The term synchronous impedance has long been used in the literature without a modifier; the term direct-axis synchronous reactance, although by no means new, is a more recently developed term.

The reactance $x_d$ may be expressed as the sum of the armature leakage reactance $x_\ell$ and the reactance $x_{ad}$ corresponding to the fundamental space wave of armature flux in the direct axis:

$$x_d = x_\ell + x_{ad} \tag{13}$$

Unless otherwise specified, $x_d$ is determined at no saturation.

**Quadrature-Axis Synchronous Reactance** $x_q$ is the ratio of the fundamental-frequency component of reactive armature voltage, due to the fundamental-frequency quadrature-axis positive-sequence component of armature current, to this component of current under sustained balanced conditions.

The following test may be applied to determine $x_q$: With the rotor rotated mechanically at synchronous speed and the field winding unexcited, let sustained quadrature-axis positive-sequence armature currents of rated frequency be applied at the machine terminals so that the center line of the fundamental mmf space wave lines up with the quadrature axis as in Fig. 4(*c*). When currents are applied from an external source through balanced reactances having a high impedance to harmonic currents, there will be no appreciable harmonic currents. Under this specified condition of operation, the ratio of the resultant fundamental-frequency component of reactive armature voltage in any phase to the current in that phase is the quadrature-axis synchronous reactance $x_q$.

The reactance $x_q$ may be expressed as the sum of the armature leakage reactance $x_\ell$ and the reactance $x_{aq}$ corresponding to the fundamental space wave of armature flux in the quadrature axis:

$$x_q = x_\ell + x_{aq} \qquad [14]$$

Unless otherwise specified, $x_q$ is determined at no saturation. (*Note:* It is well to remember that *high reactance is associated with high flux, high permeance, and flux paths in iron;* whereas low reactance is associated with flux paths largely in air.)

The air gap is longer in the quadrature axis of a salient-pole machine than in the direct axis; therefore the permeance in the quadrature axis is lower than in the direct axis, and $x_{aq}$ and $x_q$ are lower than $x_{ad}$ and $x_d$, respectively. The average ratio of $x_q$ to $x_d$ is about 0.6. (See Table I for representative values of $x_d$ and $x_q$.)

In a cylindrical-rotor machine, because of the field slots in the quadrature axis (see Fig. 2), there is a slightly longer average air gap in this axis than in the direct axis; the permeance is therefore lower than in the direct axis, and $x_q$ is slightly smaller than $x_d$. However, in most problems involving cylindrical-rotor machines, $x_d$ and $x_q$ may be assumed equal with negligible error.

**Direct-Axis Transient and Subtransient Reactances.**[5-7] Consider a salient-pole machine *without* an amortisseur or field collar. With the rotor rotated mechanically at synchronous speed and the field winding short-circuited through the exciter, let direct-axis positive-sequence armature currents of rated frequency be suddenly applied so that the fundamental mmf space wave lines up with the direct axis. The armature mmf which suddenly appears over the direct axis attempts to change the flux linkages of the field winding; but, according to Lenz's law, or the constant linkage theorem, flux linkages with a closed circuit cannot change instantly. A current is therefore induced in the field in such a direction as to oppose the armature mmf. At the first instant, this induced field current is sufficient

## TABLE I

### REPRESENTATIVE REACTANCES OF 60-CYCLE SYNCHRONOUS MACHINES IN PER CENT BASED ON RATING

| | | $x_d$ | $x_q$ | $x_d'$ | $x_q'$ | $x_d''$ | $x_q''$ | $x_0$ | $T_{d0}$ | $x_p$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Solid rotor turbine generators | 2-pole | 95–145<br>115 | 90–140<br>110 | 12–21<br>15 | | 6–12<br>9 | 8–14<br>11 | 1–7<br>3 | 1–7<br>4 | 8–15<br>12 |
| | 4-pole | 95–145<br>115 | 90–140<br>110 | 20–28<br>23 | | 10–16<br>13 | 12–18<br>15 | 1–14<br>4 | 4–10<br>6 | 10–20<br>17 |
| Salient-pole generators (with amortisseur) | Waterwheel | 60–135<br>100 | 40–85<br>65 | 20–50<br>35 | 45–85<br>65 | 15–35<br>23 | 25–45<br>30 | 5–20<br>7 | 2–10<br>5 | 15–40<br>25 |
| | Engine-driven | 65–150<br>110 | 40–100<br>70 | 15–45<br>30 | 40–100<br>70 | 15–30<br>20 | 20–40<br>30 | 2–15<br>6 | 1–4<br>2 | 15–35<br>25 |
| Condensers | | 150–220<br>180 | 100–150<br>120 | 30–60<br>40 | 100–150<br>120 | 18–35<br>30 | 20–40<br>35 | 5–20<br>8 | 5–10<br>7 | 25–45<br>35 |
| Motors | High-speed | 65–90<br>80 | 50–70<br>65 | 15–35<br>25 | 50–70<br>65 | 10–25<br>17 | 12–30<br>20 | 2–15<br>5 | 1–4<br>2.5 | 10–30<br>20 |
| | Low-speed | 80–150<br>110 | 60–110<br>80 | 20–50<br>40 | 60–110<br>80 | 15–35<br>30 | 25–40<br>35 | 4–27<br>7 | 1–4<br>2.5 | 15–45<br>35 |

NOTES: (*a*) For salient-pole generators without amortisseurs $x_q'' = x_q'$.
(*b*) $x_0$ varies from about 15% to 60% of $x_d''$, depending upon winding pitch.
(*c*) There is no very definite value of $x_q'$ for solid rotor turbine generators. It may be about 60% of $x_d$ in some cases.

to maintain constant field linkages, and the flux resulting from the suddenly applied armature currents is forced into the low-permeance paths of the interpolar spaces. The flux per ampere linking the armature due to suddenly applied currents is therefore less than that due to sustained currents. Direct-axis transient reactance, which is a measure of the armature linkages per ampere at the first instant, is considerably smaller than direct-axis synchronous reactance.

Consider next a salient-pole machine equipped with an amortisseur winding, with the rotor rotated mechanically at synchronous speed and the field winding short-circuited through the exciter. If positive-sequence direct-axis currents of rated frequency are suddenly applied at the machine terminals, currents are induced in the amortisseur in addition to those in the field winding to oppose the change in flux linkages. Since the amortisseur winding further restricts the regions where the flux produced by the armature current can go, the flux per ampere linking the armature is less than that corresponding to transient conditions. Direct-axis subtransient reactance, which is a measure of the armature linkages per ampere at the first instant, is therefore smaller than the direct-axis transient reactance of the same machine.

The term subtransient reactance was invented[5] to define the reactance associated with currents induced in rotor circuits (such as amortisseurs) which decay rapidly and to differentiate it from transient reactance, which is associated with currents induced in the field which decay slowly. As subtransient reactance is used in determining initial values of currents and voltages resulting from sudden changes in system conditions, it may be determined by applying a sudden symmetrical three-phase short circuit at the machine terminals from a condition of no-load and synchronous speed, and defined as follows.

*Direct-axis subtransient reactance* $x_d''$ is the ratio of the fundamental-frequency component of reactive armature voltage, due to the *initial* value of the fundamental-frequency direct-axis positive-sequence component of armature current, to this component of current under suddenly applied balanced load conditions. This definition applies to machines with or without amortisseur windings or their equivalents.

In a salient-pole machine without an amortisseur winding or its equivalent, direct-axis transient reactance $x_d'$ is the same as direct-axis subtransient reactance $x_d''$ and may be similarly defined. In a machine with an amortisseur winding or its equivalent, $x_d'$ may be similarly defined provided the high-decrement currents are neglected.

*Direct-axis transient reactance* $x_d'$ is the ratio of the fundamental-frequency component of reactive armature voltage, due to the fundamental-frequency direct-axis positive-sequence component of armature current, to this component of current under suddenly applied balanced load conditions, the value of current to be determined by extrapolation of the envelope of fundamental-frequency component of the current wave to the instant of sudden application of load, neglecting the high-decrement currents during the first few cycles.

In a salient-pole machine without an amortisseur or its equivalent, there will be no high-decrement current to be neglected; the above definition is therefore applicable.

The definition of $x_d'$ given here is a modified version of the definition of reference 7, with " fundamental-frequency component of the current wave " replacing " a-c component of the current wave."

The interpretation of the definitions for $x_d''$ and $x_d'$ implies that the current to be used in determining $x_d''$ is the value of the envelope of the wave of the fundamental-frequency component of short-circuit current projected to the instant of short circuit; and the current for the determination of $x_d'$ is from the same or a similar envelope with the high-decrement currents neglected. With a symmetrical three-phase short circuit at the machine terminals from a condition of no load and synchronous speed, the rated frequency component of short-circuit current may be considered a wattless current, and therefore a direct-axis current, because of the low ratio of resistance to reactance. The A.I.E.E. recognizes[7] two values of direct-axis transient and subtransient reactances, one determined at rated current and one at rated voltage. The rated current and rated voltage values of $x_d'$ and $x_d''$ will differ because of different degrees of saturation.

Oscillograms taken of currents in the three phases of a salient-pole machine with an amortisseur winding or a field collar following a sudden three-phase short circuit at the terminals of the machine at no load and rated speed will show initial a-c components of current whose magnitudes decay rapidly for a short time and then decrease slowly until sustained currents result. If a smooth curve is drawn on the oscillogram through the crests of the current waves of any phase in which there is no d-c component of current (or if the d-c component is eliminated) and projected to the vertical axis at time $t = 0$, an envelope similar to that given in Fig. 8 will be obtained. Such an envelope is that of the total a-c component of current (not the fundamental-frequency component of the a-c component of current) and therefore contains harmonic components of current. It can be used, however, to obtain approximate values of $x_d'$ or $x_d''$. To satisfy the present definitions, harmonic components of current should be eliminated before the envelope is drawn. The odd-order harmonic currents will be no more pronounced than in the determination of direct-axis synchronous reactance by a sustained three-phase short-circuit test; but even harmonic currents (discussed later) resulting from the flux trapped in the short-circuited phases, which disappear before sustained conditions are reached, may be present in subtransient and transient components of short-circuit current. This means that $x_d'$ and $x_d''$ when determined from the envelope of the a-c current wave may be influenced by the point of the voltage wave at which the short circuit is applied and also by the ratios $x_d''/x_q''$ and $x_d'/x_q'$.[8] In most cases the effect is small enough to be neglected.

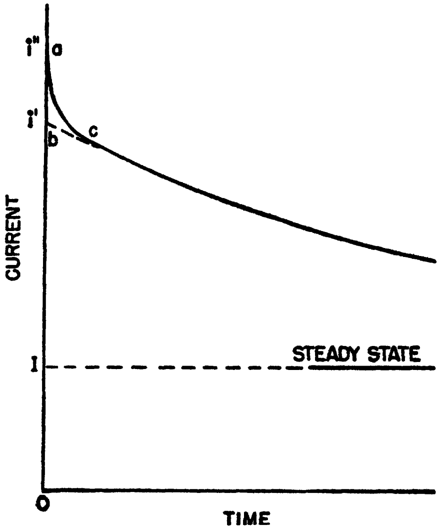

FIG. 8. Envelope of a-c component of symmetrical three-phase short-circuit current wave.

The rated current value of $x_d'$ is determined with an initial voltage such as to give a transient value of a-c short-circuit current plus the sustained value equal to rated current. The rated-voltage value of $x_d'$ is determined with an initial voltage equal to rated voltage. For both values, $x_d'$ is the ratio of the fundamental-frequency component of no-load voltage to the a-c component (or more exactly, the fundamental-frequency a-c component) of short-circuit current obtained by extrapolation of the envelope of the a-c component of the armature current wave to the instant of the sudden application of the three-phase short circuit, neglecting the high-decrement currents during the first few cycles.

In Fig. 8, $i''$ at *a* is the initial per unit a-c current; $i'$ at *b* is the per unit a-c current determined by extrapolation of the envelope to the instant of application of load neglecting the high-decrement current during the first few cycles; *I* is the sustained current.

In determining $x_d'$, the current $i'$ at *b* in Fig. 8, which is the sum of the sustained current *I* and the transient component $i'-I$, is the current specified. For the rated current value of $x_d'$ the current $i'$ must equal rated current; for the rated voltage value of $x_d$ the current $i'$ must be that obtained when the initial no-load voltage is rated voltage.

In determining $x_d''$ by a sudden three-phase short-circuit test at the terminals of a machine at rated speed, the current used for the rated current value is the same as that used to determine the rated current value of direct-axis transient reactance already described. The rated voltage value is that obtained with a no-load voltage equal to rated voltage. In both cases, $x_d''$ is determined as the ratio of the per unit no-load voltage to the initial value of the per unit a-c component (more exactly, the fundamental-frequency component of the a-c component) of short-circuit current. In Fig. 8, the current specified is $i''$ at $a$.

**Quadrature-Axis Transient Reactance $x_q'$ and Subtransient Reactance $x_q''$.** Consider a salient-pole machine without an amortisseur winding or its equivalent in the quadrature axis. With the rotor rotated mechanically at synchronous speed, let positive-sequence quadrature-axis currents of rated frequency be suddenly applied. The voltages induced in one-half of the field winding or in amortisseur winding in the direct axis balance those induced in the other half, so that no currents are induced in the rotor circuits. The flux paths under this condition of operation for a salient-pole machine are the same as those for sustained conditions.

*In a salient-pole machine without an amortisseur winding in the quadrature axis*, quadrature-axis subtransient, transient, and synchronous reactances are equal:

$$x_q'' = x_q' = x_q = x_l + x_{aq} \qquad [15]$$

*In a salient-pole machine with an amortisseur winding or its equivalent in the quadrature axis*, quadrature-axis transient reactance may still be taken equal to quadrature axis synchronous reactance.

$$x_q' = x_q \qquad [16]$$

In a cylindrical-rotor machine, the effect of the currents induced in the solid rotor in modifying the permeance, after subtransient effects have disappeared, will make the quadrature-axis transient reactance $x_q'$ less than the quadrature-axis synchronous reactance $x_q$ but greater than the subtransient quadrature-axis reactance $x_q''$ which is defined below.

With rated positive-sequence quadrature-axis current of fundamental frequency suddenly applied to a machine equipped with an amortisseur winding in the quadrature axis and rotated at synchronous speed, currents are induced in the amortisseur which oppose the change of flux linkages. The permeance of the flux paths and the armature linkages per ampere are therefore less at the first instant than under transient conditions, and $x_q''$ is smaller than $x_q'$.

*Quadrature-axis subtransient reactance* $x_q''$ is the ratio of the fundamental-frequency component of reactive armature voltage, due to the initial value of the fundamental-frequency quadrature-axis positive-sequence component of armature current, to this component of current under suddenly applied balanced load conditions.

In a cylindrical-rotor machine, the solid rotor and the wedges over the field winding produce an effect similar to that of an amortisseur winding in the quadrature axis.

**Reactances Variable Quantities.** Except for $x_d$ and $x_q$ without saturation, it is evident from the above definitions and discussions that positive-sequence reactances are variable quantities which depend upon the way they are defined and the degree of saturation at which they are determined by tests. In synchronous reactances, saturation is in the iron of the main flux paths; in transient and subtransient reactances, saturation is in the leakage paths which are partly in iron and partly in air. In problems involving reactances of synchronous machines, where a high degree of precision is not required in these reactances, the representative values given in Table I may be used when exact values are not available.

## Reciprocal Per Unit Equivalent Circuits for Armature and Rotor Circuits

In the previous work of this chapter, base field current has been defined as the field current required to produce rated voltage at rated speed on the air-gap line. This choice of base field current is convenient for constructing vector diagrams, but it is not convenient for including the effects of currents in rotor circuits additional to the field circuit. This can best be done by means of equivalent circuits in which the equivalent mutual reactances between stator and rotor circuits in the direct or quadrature axis are reciprocal. Several different per unit systems have been used which are reciprocal. It has been suggested,[9] however, that a preferred reciprocal system be adopted in which base field current is so defined that $x_{ad}$, the reactance of direct-axis armature reactance, is the equivalent per unit mutual reactance between armature and field. Base currents in all other rotor circuits are analogously defined, so that the equivalent per unit mutual reactances between armature and rotor circuits in the direct and in the quadrature axis are reciprocal. The widely used base or unit armature quantities previously defined in this chapter are retained. In addition to defining base rotor current, it is necessary to establish an equivalent ratio of turns for armature and rotor circuits. The construction of a reciprocal per unit reactance diagram for armature and field will be illustrated. (For a comprehensive treatment of the equivalent circuits of synchronous machines see reference 9.) Let base field current be indicated by $I_{fdo}$, where

$I_{fdo}$ = that field current in amperes which produces on the air-gap line at rated speed a per unit voltage in each armature phase equal to per unit $x_{ad}$.

Base field current $I_{fdo}$ can be read directly on the air-gap line of the no-load saturation curve corresponding to a per unit armature voltage equal to per unit $x_{ad}$. Let $i_{ao}$ indicate base armature current, where $i_{ao}$ = peak value of rated armature current in amperes.

The fundamental mmf space wave resulting from positive-sequence direct-axis armature currents is stationary in the direct axis. Its amplitude is $\frac{3}{2}$ the amplitude of the pulsating fundamental mmf space wave in each phase (see Appendix B). In a machine with a sinusoidally distributed armature winding, the fundamental mmf space wave due to unit armature currents has an amplitude of $(\frac{3}{2})i_{ao}$. The mmf wave produced by rated armature currents has been chosen as the base or unit mmf space wave in defining unit armature quantities. An armature current of $(\frac{3}{2})i_{ao}$ in the direct axis produces the same armature voltage as $I_{fdo}$ defined above, each producing a per unit armature voltage equal to per unit $x_{ad}$. The ratio of base currents in armature and field is therefore $(\frac{3}{2})i_{ao}$ to $I_{fdo}$. With this ratio of base currents, the equivalent ratio of armature turns to field turns is $I_{fdo}$ to $(\frac{3}{2})i_{ao}$. This ratio of turns determines the ratio of base voltages in armature and field circuits.

In constructing a reciprocal per unit equivalent circuit in the direct axis, it is convenient to represent one phase of the armature with rated armature current as base current rather than $(\frac{3}{2})i_{ao}$ as base current. In an equivalent circuit so represented, base field kva is $(\frac{3}{2})$ times base peak armature kva. Field resistance and self-inductance can be expressed in per unit with base field quantities; or they can be multiplied by $\frac{3}{2}$ times the square of the equivalent ratio of turns, that is, by $\frac{3}{2}[I_{fdo}/(\frac{3}{2}i_{ao})]^2$, to refer them to the armature, then divided by their respective base armature quantities.

The per unit field leakage reactance $x_{f\ell}$ is the difference between field per unit self-reactance $x_{ffd}$ and the mutual reactance $x_{ad}$. The per unit armature leakage reactance $x_\ell$ is $x_d - x_{ad}$.

**Equivalent Circuits in the Direct and Quadrature Axes.** If field quantities are expressed in a reciprocal per unit system,[9] where per unit $x_{ad}$ is mutual to armature and field, direct-axis transient reactance $x_d'$ may be determined from the equivalent circuit of Fig. 9(*a*), where $x_{f\ell}$ is field leakage reactance, $x_{ffd}$ is total field reactance, $x_\ell$ is armature leakage reactance, and $x_{ad}$ is mutual reactance between armature and field. Direct-axis transient reactance $x_d'$, which is a reactance viewed from the armature terminals, is

$$\begin{aligned} x_d' &= x_\ell + \frac{x_{ad}x_{f\ell}}{x_{ad} + x_{f\ell}} = x_\ell + \frac{x_{ad}(x_{ffd} - x_{ad})}{x_{ffd}} \\ &= x_d - \frac{x_{ad}^2}{x_{ffd}} \end{aligned} \qquad [17]$$

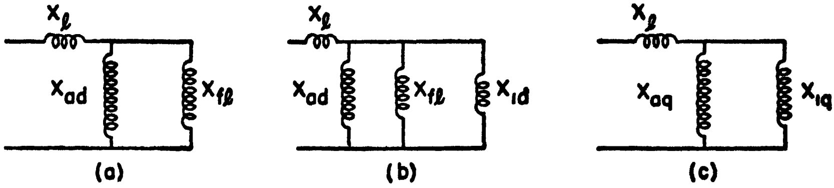

Fig. 9. (*a*), (*b*), and (*c*) Reciprocal per unit equivalent circuits for armature and rotor circuits for determining $x_d'$, $x_d''$, and $x_q''$, respectively.

It will be noted that direct-axis synchronous reactance $x_d$ is obtained by opening the field leakage branch $x_{f\ell}$ of Fig. 9(*a*). In steady state, the induced field current resulting from suddenly applied current has decayed to zero, and direct-axis synchronous reactance $x_d = x_\ell + x_{ad}$ is offered to the flow of sustained direct-axis armature currents.

Direct-axis subtransient reactance $x_d''$ may be determined from the equivalent circuit of Fig. 9(*b*), in which the amortisseur winding in the direct axis is represented as an equivalent single winding having a leakage reactance $x_{1d}$. This circuit, based on the assumption that the mutual reactances between armature and field, between armature and amortisseur, and between amortisseur and field are all equal to $x_{ad}$, is reasonably accurate for many machines. The direct-axis subtransient reactance $x_d''$ given by Fig. 9(*b*) is

$$x_d'' = x_\ell + \cfrac{1}{\cfrac{1}{x_{ad}} + \cfrac{1}{x_{1d}} + \cfrac{1}{x_{f\ell}}} \qquad [18]$$

The quadrature-axis subtransient reactance $x_q''$ may be determined from the per unit reciprocal equivalent circuit of Fig. 9(*c*), in which $x_{aq}$ is the mutual reactance between armature and rotor circuits in the quadrature axis and $x_{1q}$ represents the equivalent leakage reactance of the amortisseur winding of a salient-pole machine or the analogous effect in a cylindrical-rotor machine. Viewed from the armature terminals,

$$x_q'' = x_\ell + \frac{x_{1q}x_{aq}}{x_{1q} + x_{aq}} \qquad [19]$$

In a salient-pole machine, after currents induced in the amortisseur winding have died out, quadrature-axis transient and synchronous reactance $x_q'$ and $x_q$ can be evaluated by opening the reactance branch $x_{1q}$; then,

$$x_q' = x_q = x_\ell + x_{aq} \qquad [20]$$

**Negative-Sequence Reactance $x_2$.** Negative-sequence reactance is here defined as the ratio of the fundamental-frequency reactive armature voltage, due to the fundamental-frequency negative-sequence component of armature current, to this component of current with the rotor rotating at synchronous speed in its normal direction of rotation and the field winding short-circuited through the exciter.

The rotating space waves of mmf produced by rated-frequency armature currents of negative-sequence phase order differ from those due to positive-sequence phase order only in their direction of rotation. The fundamental space wave of mmf rotates at synchronous speed with respect to the stator but in a direction opposite to the normal direction of rotation of the rotor. It therefore rotates backward at twice synchronous speed with respect to the rotor. As this mmf wave passes rapidly over each axis of the poles, the permeance presented varies rapidly from that in the direct to that in the quadrature axis, the reactance corresponding alternately to direct- and quadrature-axis subtransient reactances.

An equation for negative-sequence reactance $x_2$, as defined here, is derived in the following chapter. From Chapter VIII, [65],

$$x_2 = \tfrac{1}{2}(x_d'' + x_q'') \qquad [21]$$

An equation for the third harmonic positive-sequence voltage resulting from the flow of negative-sequence currents is also given. (See Chapter VIII, [63].)

In a salient-pole machine without an amortisseur winding or its equivalent, $x_d''$ is replaced by $x_d'$ and $x_q''$ by $x_q' = x_q$.

**Subtransient, Transient, and Sustained Negative-Sequence Reactances.** Most system studies are made with the rotor running at or near synchronous speed; therefore negative-sequence reactance is taken to mean that reactance presented by the machine to the flow of fundamental-frequency negative-sequence currents, with the rotor running at synchronous speed in its normal direction of rotation. Actually, this reactance is not greatly different at the same degree of saturation in $x_d''$ and $x_q''$ for all rotor speeds from synchronous down to standstill. As $x_2$ is determined from $x_d''$ and $x_q''$, it is the same under subtransient, transient, and sustained conditions.

**Second Harmonic Field Currents.** With negative-sequence currents in the armature, the fundamental mmf space wave rotating at twice synchronous speed relative to the rotor will give rise to a second harmonic voltage in the field which, being short-circuited through the exciter, will produce second harmonic currents in the field. Oscillograms of the field current in both salient-pole and cylindrical-rotor machines show this second harmonic component. It is small in a cylindrical-rotor machine because of the shielding effect of the slot wedges and body of the rotor. It is limited in a salient-pole machine by the high reactance of the field to second harmonic current, and by the shielding effect of the amortisseur winding (if present) in the direct axis. The field flux produced by second harmonic field current is a fundamental space wave pulsating at twice synchronous speed relative to the rotor, which can be resolved into two waves, one rotating forward and one backward with respect to the rotor at twice synchronous speed. Relative to the armature, one rotates backward at synchronous speed and the other forward at three times synchronous speed. One induces negative-sequence reactive armature voltages, the other positive-sequence third harmonic armature voltages; but these voltages are taken into account in determining the impedance to negative-sequence armature currents and the induced third harmonic armature voltage in terms of $x_d''$ and $x_q''$, which include the effects of induced rotor currents.

**Zero-Sequence Reactance $x_0$.** Zero-sequence reactance is the ratio of the fundamental-frequency component of reactive armature voltage, due to the fundamental-frequency zero-sequence component of armature current, to this component of current.

If zero-sequence currents of fundamental frequency are applied to a three-phase synchronous machine, the mmf space waves of the three phases pulsate sinusoidally in time at impressed frequency. These space waves are the same as those for positive-sequence currents of fundamental frequency and can be similarly resolved into a fundamental space wave and odd-harmonic space wave of mmf cosinusoidally distributed about the axes of magnetization of the three phases. The equations for these space mmf waves for phases $a$, $b$, and $c$ are given by the terms in brackets in [B–1]–[B–3], Appendix B. As the waves in all phases pulsate in phase, the multipliers cos $(t - 120)$ and cos $(t + 120)$ of [B–2] and [B–3] will be replaced by cos $t$. Then, when the equations are added, it will be seen that the fundamental and all harmonic space mmf waves, except the third and multiples of the third, which pulsate at fundamental frequency in the air gap and ends of the machine, disappear. The slot leakage flux and the third harmonic mmf space wave are very sensitive to coil pitch. For a full-pitch coil, the currents in the top and bottom conductors of any slot are in the same direction, and the slot leakage reactance term is the same as if positive- or negative-sequence currents were impressed. At other than full pitch, the coil sides in some of the slots carry currents in opposite directions and the slot, air-gap, and end-leakage terms are all smaller. The minimum value occurs at $\frac{2}{3}$ pitch where the currents in the two coil sides of every slot are in opposition. At this pitch, the slot leakage is a minimum, and the third harmonic mmf space wave in the air gap and end turns disappears. Thus, zero-sequence reactance may vary greatly, depending upon the coil pitch, but it is never greater than positive-sequence armature leakage reactance and may be less than half of it. Because there is no fundamental space wave of armature mmf, zero-sequence reactance is practically independent of rotor motion, and is the same under subtransient, transient, and steady-state conditions.

Table I gives typical values of fundamental-frequency reactances to positive-, negative-, and zero-sequence currents. As defined, fundamental-frequency reactances do not include the presence of harmonic armature voltages resulting from fundamental-frequency currents, whether of positive, negative, or zero sequence. These reactances may be used in the calculation of *fundamental-frequency* currents and voltages even if harmonic currents and voltages are known to be of importance, provided the effects of harmonic currents in producing fundamental-frequency currents and voltages can be neglected. Harmonic armature voltages and currents are discussed in the following chapter.

**Resistances.** The d-c armature resistance per phase is indicated by $r_a$, the a-c armature resistance per phase by $r$, the field resistance by $R_f$. These resistances depend upon temperature.

D-c armature resistances per phase in ohms, as usually furnished, are given at stated machine temperatures. Expressed in per cent on the machine rating, they vary from about 0.1 to 1.0 per cent. As an approximation, the positive-sequence a-c resistance may be taken as 50% above the d-c value to include hysteresis and eddy current losses. The positive-sequence resistance is thus very low relative to the reactance and in many system problems can be neglected with negligible error. The zero-sequence resistance may be assumed to be equal to the positive-sequence resistance. The negative-sequence resistance is higher than the positive because of the greater $I^2R$ loss in the rotor circuits due to the backward-rotating fundamental flux wave.

In circuit calculations involving relationship between currents and voltages, positive-, negative-, and zero-sequence resistances ($r_1$, $r_2$, and $r_0$) as discussed above are the ratios of the fundamental-frequency *in-phase* components of armature voltage, due to the fundamental-frequency positive-, negative-, and zero-sequence components, respectively, of armature current, to these components of current. The positive- and zero-sequence resistances thus defined are used in calculating losses in a machine due to flow of positive- and zero-sequence currents. With negative-sequence currents, however, part of the loss is that required to overcome the negative-sequence braking torque.

**Negative-Sequence Braking Torque and Negative-Sequence Resistance Used in Calculating Losses.** The fundamental space mmf wave, resulting from fundamental-frequency negative-sequence currents, rotating at synchronous speed in the opposite direction to that of the rotor, creates a torque in the negative direction called the negative-sequence braking torque.

With the synchronous machine considered an induction machine of zero slip currents, and $r_p(= r_1)$ and $r_s$ representing stator and rotor resistances, respectively, the negative sequence power loss *due to current flow* from the negative-sequence equivalent circuit of Fig. 2(*d*) of Chapter VI with $s = 0$ is approximately

$$|I_{a2}|^2 r_2 = |I_{a2}|^2 \left(r_1 + \frac{r_s}{2}\right) \qquad [22]$$

where $r_2$ is the negative-sequence resistance and $r_1$ the positive-sequence armature resistance used in circuit calculations discussed above.

From [22],

$$r_s = 2(r_2 - r_1) \qquad [23]$$

To the power loss in [22], the power at synchronous speed required to overcome the negative sequence torque $|I_{a2}|^2(r_s/2)$ given by [10] of Chapter VI with $s = 0$ must be added. With $r_s$ replaced by its value given in [23], the total negative-sequence power loss is approximately

$$|I_{a2}|^2 \left(r_2 + \frac{r_s}{2}\right) = |I_{a2}|^2(r_1 + r_s) = |I_{a2}|^2(2r_2 - r_1) \qquad [24]$$

$2r_2 - r_1$ is the resistance used to calculate the negative-sequence loss in a synchronous machine. As previously stated, in circuit calculations involving relationships between currents and voltages $r_2$ is used.

As all resistances associated with modern synchronous machines are small, except resistance in the neutral or other resistances inserted for special purposes, in many system problems involving only fundamental-frequency components (short-circuit calculations, for example) the resistances of a synchronous machine may be neglected with but slight error because of their low values relative to the reactances.

**Synchronous Machines as Induction Machines.** Most synchronous machines depend upon inductive action for more completely satisfying the requirement of good synchronous operation. For example, synchronous motors operate as induction motors while being brought up to speed so they can be synchronized. When synchronized, synchronous motors and generators depend upon their inherent induction characteristics to provide damping torque which allows them to operate stably without hunting.

## Diagrams of Synchronous Machines — Saturation Neglected

Figure 10(*a*) shows the diagram of a salient-pole machine under normal balanced operation with saturation neglected, and all fundamental-frequency voltages which contribute to armature terminal voltages superposed. In this diagram, all quantities are in per unit of their respective base values. This makes it possible to represent mmf's, flux linkages, voltages, and currents on the same diagram. The position of the direct axis is indicated as reference, and all quantities are referred to the rotor. $I_f$ is the d-c field current in per unit of that base field current which produces unit fundamental-frequency armature voltage on open circuit with saturation neglected. In per unit, $I_f = \psi_f =$ the per unit fundamental flux linking the armature due to field current alone. The fundamental-frequency armature voltage generated by this field flux, which rotates with the rotor at synchronous speed, is shown 90° behind the field flux and is indicated by $e_q$. It will be noted that the voltage $e_q$ corresponding to field current $I_f$ appears in the quadrature axis. The use of $e_q$ to represent the voltage due to the rotation of the field flux follows the notation of reference 10. With $I_f$ defined in per unit as above and saturation neglected, $e_q$ in per unit of rated armature voltage is equal in magnitude to $I_f$ in per unit of base field current. The voltage $e_q$ is called the internal generated voltage or the excitation voltage of the machine.

The mmf due to positive-sequence armature current magnetizes in the direction indicated by $i$. The length of $i$ is equal to the per unit positive-sequence armature current; $i_d$ and $i_q$ are the per unit components of $i$ which magnetize in the direct and quadrature axes, respectively. In Fig. 10($a$), $i_d$ and $i_q$ are both positive (see Fig. 5), indicating an overexcited generator supplying positive power, as previously stated. The reactances $x_d$ and $x_q$ in per unit of base impedance when multiplied by $i_d$ and $i_q$, respectively, and turned backward 90° represent the per unit fundamental-frequency reactive voltages induced in the quadrature and direct axes, respectively, under sustained conditions. The per unit flux linkages $i_dx_d$ and $i_qx_q$ in phase with $i_d$ and $i_q$, respectively, are not indicated on the diagram. The per unit flux $i_dx_d$ opposes the field flux $\psi_f$ when $i_d$ is positive; but it will be in the same direction as $\psi_f$ in an underexcited machine where $i_d$ is negative.

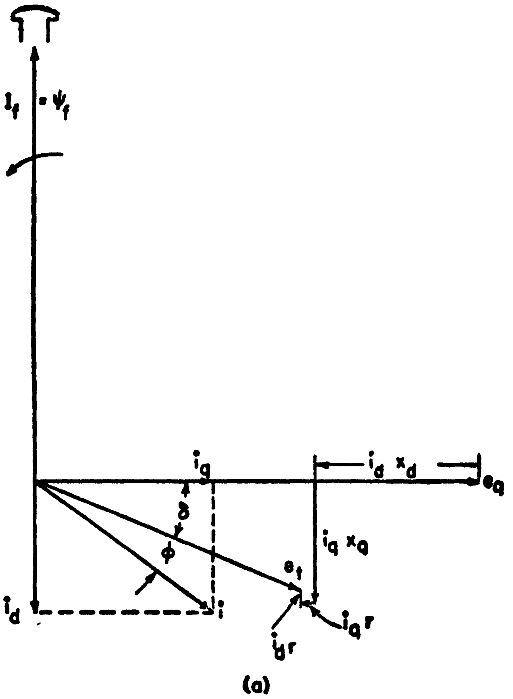

FIG. 10($a$). Per unit vector diagram of salient-pole machine under sustained balanced operation with generated voltage $e_q$ as reference vector.

In Fig. 10($a$), $i_d$ (shown positive) magnetizes in a direction opposite to that of $I_f$; the reactive voltage $i_dx_d$ appearing in the quadrature axis is therefore subtracted from $e_q$. $i_q$ (shown positive) magnetizes 90° ahead of $i_d$; the reactive voltage $i_q x_q$, 90° ahead of the voltage $i_d x_d$, is a direct-axis voltage. When the per unit resistance voltage drops $ri_d$ and $ri_q$ in the direct and quadrature axes, respectively, are also subtracted, the resultant per unit terminal voltage $e_t$ is obtained. The angle $\phi$ between $i$ and $e_t$ is the power-factor angle; $\phi$ is positive if $i$ leads $e_t$ but negative if $i$ lags $e_t$. In Fig. 10($a$), $\phi$ is negative. $\delta$ is the angle by which $e_q$ leads $e_t$.

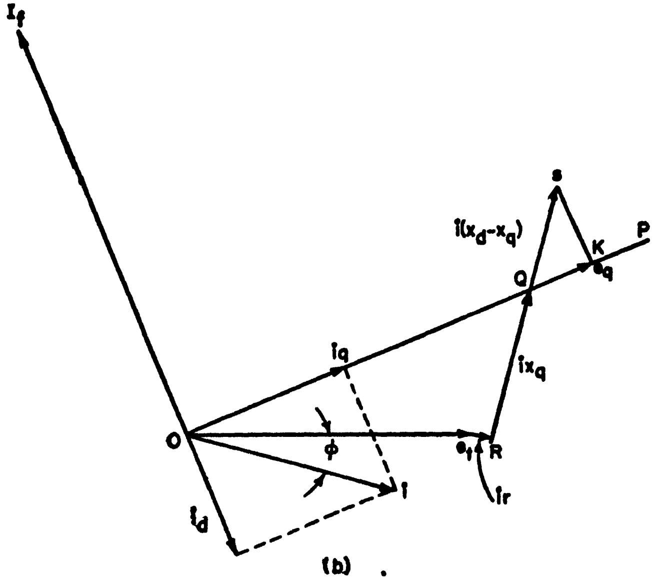

FIG. 10($b$). Per unit diagram of salient-pole machine under sustained balanced operation with terminal voltage $e_t$ as reference vector.

When positive-sequence current, terminal voltage, and power factor are given, Fig. 10($b$) indicates the procedure in constructing the per unit diagram of a salient-pole generator under normal balanced operation to determine per unit field current $I_f$. The per unit terminal voltage $e_t$ is laid off as reference vector; the per unit current $i$ at the power-factor angle $\phi$ is drawn. The per unit resistance voltage drop $ir$ is added to $e_t$ in the direction of $i$ and terminated at $R$. The per unit reactive voltage drop $ix_q$ is added at $R$, 90° ahead of $ir$ and terminating at $Q$. Through the origin $O$ and $Q$, the straight $OQP$ is drawn. This is the quadrature axis. At $Q$ the reactance voltage drop $i(x_d - x_q)$ is added in line with $RQ$ and terminated at $S$. From $S$ a perpendicular is dropped to $OQP$, intersecting it at $K$. Then $OK = e_q =$ per unit voltage generated by the rotation of the flux produced by field current $I_f$ acting alone = excitation voltage. The direct axis is drawn perpendicular to $OQP$, and the per unit field current $I_f$ equal to $e_q$ in magnitude and 90° ahead of $e_q$ is laid off. The projection of $i$ on the direct axis is $i_d$; the projection of $i$ on the quadrature axis is $i_q$. It will be noted that Fig. 10(*b*) differs from Fig. 10(*a*) only in that the terminal voltage $e_t$ is horizontal in Fig. 10(*b*), whereas the excitation voltage $e_q$ is horizontal in Fig. 10(*a*).

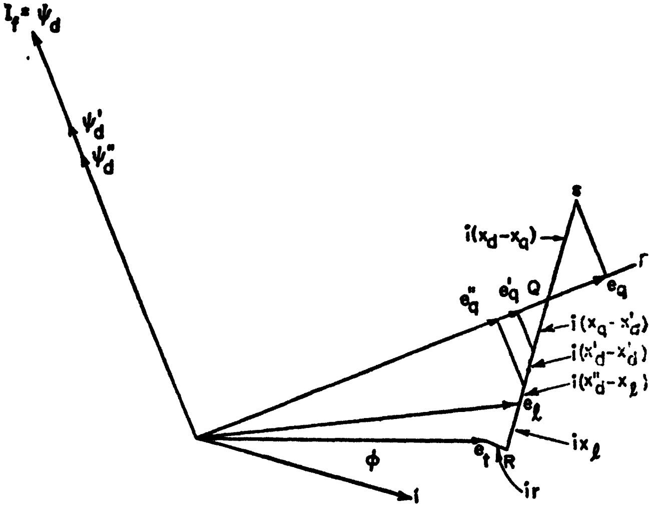

Fig. 10(*c*). Per unit vector diagram of salient-pole machine under sustained balanced operation with voltages $e'_q$ and $e''_q$ behind $x'_d$ and $x''_d$, respectively, their corresponding flux linkages in the direct axis.

Figure 10(*c*) is similar to Fig. 10(*b*) except that $e'_q$ and $e''_q$, the voltages behind direct-axis transient and subtransient reactances $x'_d$ and $x''_d$, respectively, as well as the voltage $e_q$ behind synchronous reactance $x_d$, are indicated on the quadrature axis $OQP$. $e_\ell$ indicates the voltage behind leakage reactance $x_\ell$. The flux linkages $\psi''_d$ and $\psi'_d$ in the direct axis which remain constant at the first instant following a disturbance, with and without an amortisseur winding, respectively, are indicated in Fig. 10(*c*), 90° ahead of $e''_q$ and $e'_q$, respectively.

**Cylindrical-Rotor Machine.** The cylindrical-rotor machine is a special case of a salient-pole machine, in which $x_d = x_q$ to a close approximation. In Fig. 11(*a*), the excitation voltage $e$ required to give the terminal voltage $e_t$, with current $i$ in the machine at the power factor angle $\phi$, is obtained by adding the in-phase voltage drop $ir$ and the reactive voltage drop $ix_d = ix_q$ to $e_t$, as indicated. In a machine without saturation, the required per unit field current $I_f$ is 90° ahead of $e$ but equal to $e$ in magnitude. The per unit flux $\psi_f$ mutual to armature and field, produced by field current alone, is a direct-axis flux equal to $I_f$ in magnitude when saturation is neglected.

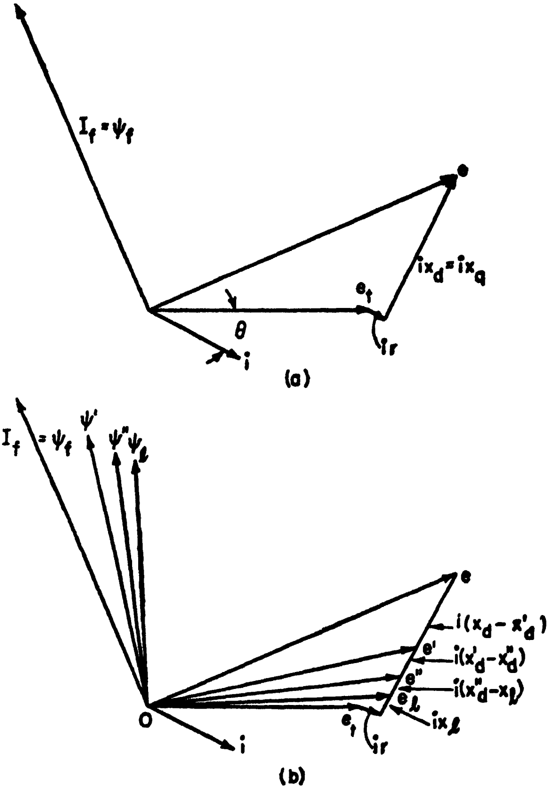

FIG. 11. Per unit vector diagrams of a cylindrical-rotor machine. (*a*) Under sustained balanced operation. (*b*) Voltages and flux linkages behind $x_l$, $x''_d = x''_q$, and $x'_d = x'_q$.

Figure 11(*b*) is the diagram of a machine in which $x_d = x_q$, $x'_d = x'_q$, and $x''_d = x''_q$. The voltages $e$, $e'$, and $e''$, and $e_l$ behind $x_d$, $x'_d$, $x''_d$, and $x_l$, respectively, are indicated. The per unit flux linkages which remain constant at the first instant during subtransient and transient disturbances are indicated by $\psi''$ and $\psi'$, respectively. The per unit flux linkages $\psi''$ and $\psi'$ which have components in both the direct and quadrature axes are 90° ahead of $e''$ and $e'$, respectively, and equal to them in magnitude.

## Relation between Field Current and Armature Terminal Voltage — Saturation Included

In the diagrams of Figs. 10 and 11 with saturation neglected, the excitation voltage $e_q$, the flux linkages $\psi_f$ due to field current $I_f$ acting alone, and the field current $I_f$ are all equal in magnitude when expressed in per unit of their respective base values. Reference to Fig. 6 will show that even with the armature on open circuit, the per unit field current required to give unit armature voltage without saturation, read on the air-gap line, is substantially less than with saturation included, read on the open-circuit saturation curve. When a machine is operated under load, the effect of saturation in increasing the field current required to give rated terminal voltage may be much more pronounced. An approximate method of including saturation in calculating field current under load is based on the use of Potier reactance $x_p$ and the saturation in the machine at $V_p$, the voltage behind $x_p$.

Consider first a machine operated at rated armature voltage and rated current at zero power factor (overexcited). In Fig. 7, $I_f$ is the total per unit field current required to give rated terminal voltage with rated zero-power-factor current in the armature. If the load is removed, the open-circuit armature voltage is $E$. Under load, the field current $I_f$ consists of three parts: (1) $I_{ft}$, the field current corresponding to rated terminal voltage $V_t$ on the air-gap line, (2) $I_{fz}$, the field current required to force zero-power-factor rated current through the synchronous impedance $Z$ at zero terminal voltage, and (3) $I_{fs}$, the additional field current required because of saturation: $I_f = I_{ft} + I_{fz} + I_{fs}$. In Fig. 7, $I_f = TB$, $I_{ft} = TF$; $I_{fz} = OS = AB$; and therefore $I_{fs} = FA$. $KC$, which is equal and parallel to $FA$, indicates the increase in field current required because of saturation. In Fig. 7, $DC$, the voltage drop in the Potier reactance added to the terminal voltage $V_t$, gives the voltage $V_p$. The voltage $V_p$ is the voltage behind the Potier reactance $x_p$. $V_p$ determines the degree of saturation in the machine and the additional field current $I_{fs}$ required because of saturation. The additional per unit field current is the distance between the air-gap line and the no-load saturation curve, both corresponding to $V_p$.

Potier reactance, unless otherwise specified, is determined from the no-load saturation curves and the zero-power-factor curve, as previously described and illustrated in Fig. 7. In the range of power factors between zero power factor (overexcited) and unity this value of Potier reactance (see Table I) will give field currents which agree approximately with test values of field currents and are of sufficient precision for many practical applications. The procedure in calculating field current $I_f$ by the use of Potier reactance when the terminal voltage $V_t$, the positive-sequence current $I_{a1}$, and the power-factor angle $\phi$ are given is as follows.

1. With all quantities in per unit and $V_t$, the terminal voltage, as reference vector, calculate the magnitudes of the vector voltages $V_p$ and $E'$, where

$$V_p = V_t + I_{a1}Z_p = V_t + |I_{a1}| (\cos\phi \pm j\sin\phi)(r + jx_p) \qquad [25]$$

$$E' = V_t + I_{a1}Z_1 = V_t + |I_{a1}| (\cos\phi \pm j\sin\phi)(r + jx_d) \qquad [26]$$

sin $\phi$ is positive for leading currents (underexcited machine) and negative for lagging currents (overexcited machine). With saturation neglected, $E'$ is the per unit excitation voltage of the machine which is equal in magnitude to $I'_f$, the per unit field current. With saturation included, $I_f$ has a higher per unit value.

2. Read the per unit field current corresponding to $V_p$ on the air-gap line and on the open-circuit saturation curve and find their difference $\Delta I_f$. The per unit field current $I_f$ is the sum of $I'_f$ (= per unit $E'$) and $\Delta I_f$.

$$I_f = I'_f + \Delta I_f = |E'| + \Delta I_f \qquad [27]$$

Problem 1 illustrates this method of calculating field current; it also shows how relations between terminal voltage and current at a specified power factor can be determined when field current is known.

**Problem 1.** In a separately excited synchronous generator, $x_d = 1.20$, $x_p = 0.20$, and $r = 0.05$ based on its rating. The no-load saturation curve is shown in Fig. 12. Determine the field current at rated voltage, 0.8 power factor and (*a*) rated current, (*b*) 150% current. The field current corresponding to 150% current at 0.8 power factor is maximum field corresponding to exciter ceiling voltage. (*c*) With field current constant at exciter ceiling, plot generator terminal voltage versus generator line current at 0.8, 1.0, and 0 power factors under balanced operation.

*Solution.* The broken curve in Fig. 12 gives $\Delta I_f$, the per unit field current required to overcome saturation at a voltage $V_p$ behind Potier reactance $x_p$. Thus if $V_p = 1.10$, the field current from the open-circuit saturation curve is 1.47. The field current on the air-gap line corresponding to $V_p = 1.10$ is 1.10. $\Delta I_f$ at $V_p = 1.10$ is therefore $1.47 - 1.10 = 0.37$. This value of $\Delta I_f$ is plotted versus 1.47 field current. To read $\Delta I_f$, find the intersection of $V_p$ with the no-load saturation curve, drop vertically down to the $\Delta I_f$ curve, and read $\Delta I_f$ on the right-hand vertical scale.

(*a*) From [25] and [26], with $V_t = 1.0$ as reference vector,

$$|V_p| = |1.0 + 1.0(0.8 - j0.6)(0.05 + j0.20)| = |1.16 + j0.13| = 1.17$$

$$|E'| = |1.0 + 1.0(0.8 - j0.6)(0.05 + j1.20)| = |1.76 + j0.93| = 1.99$$

With $|V_p| = 1.17$, from Fig. 12, $\Delta I_f = 0.55$.

$$I_f = |E'| + \Delta I_f = 1.99 + 0.55 = 2.54$$

If saturation were neglected the field current would be $I_f' = |E'| = 1.99$. With saturation included it is 2.54.

(*b*)

$$|V_p| = |1.0 + 1.5(0.8 - j0.6)(0.05 + j0.20)| = |1.24 + j0.195| = 1.26$$

$$|E'| = |1.0 + 1.5(0.8 - j0.6)(0.05 + j1.20)| = |2.14 + j1.395| = 2.55$$

With $V_p = 1.26$, $\Delta I_f = 0.93$

$$I_f = |E'| + \Delta I_f = 2.55 + 0.93 = 3.48$$

The field current $I_f = 3.48$ corresponds to exciter ceiling voltage.

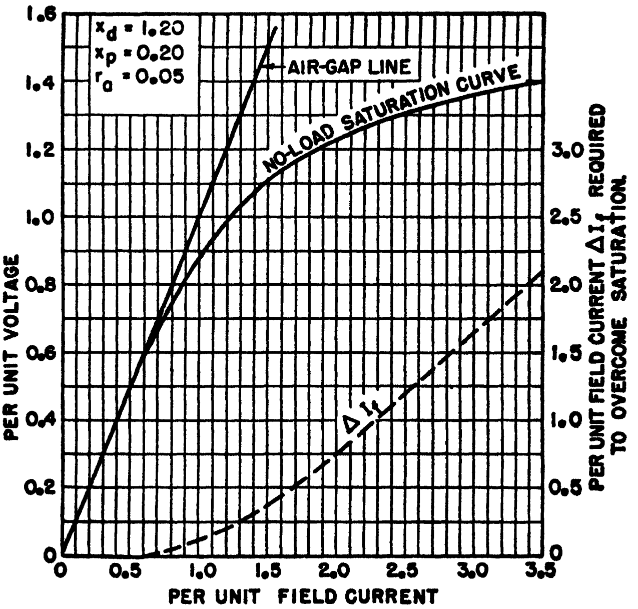

FIG. 12. No-load saturation curve for three-phase, 0.8 power-factor, separately excited generator.

(*c*) Figure 13 shows generator per unit terminal voltage versus per unit line current at power factors of 1.0, 0.8, and 0 at the generator terminals under balanced operation, with a constant per unit field current $I_f = 3.48$. These curves were determined as follows.

With zero current, the generator is on open circuit. The generator terminal voltage $V_t$, read from the no-load saturation curve, corresponding to $I_f = 3.48$, is $V_t = 1.39$. The point $I_{a1} = 0$, $V_t = 1.39$, which is common to curves of all power factors, is shown in Fig. 13 at *A*.

With zero terminal voltage, there is little if any saturation. With no saturation,

$$|E'| = |I_{a1}||0.05 + j1.20| = 1.201|I_{a1}| = I_f = 3.48$$

$$|I_{a1}| = \frac{3.48}{1.20} = 2.90$$

$$|V_p| = |I_{a1}||0.05 + j0.20| = 2.90 \times 0.206 = 0.60$$

From Fig. 12 with $V_p = 0.60$, there is no appreciable saturation, and $\Delta I_f = 0$. Therefore $|I_{a1}| = 2.90$. The point $I_{a1} = 2.90$, $V_t = 0$, which is a second point common to all curves, is shown in Fig. 13 at $B$.

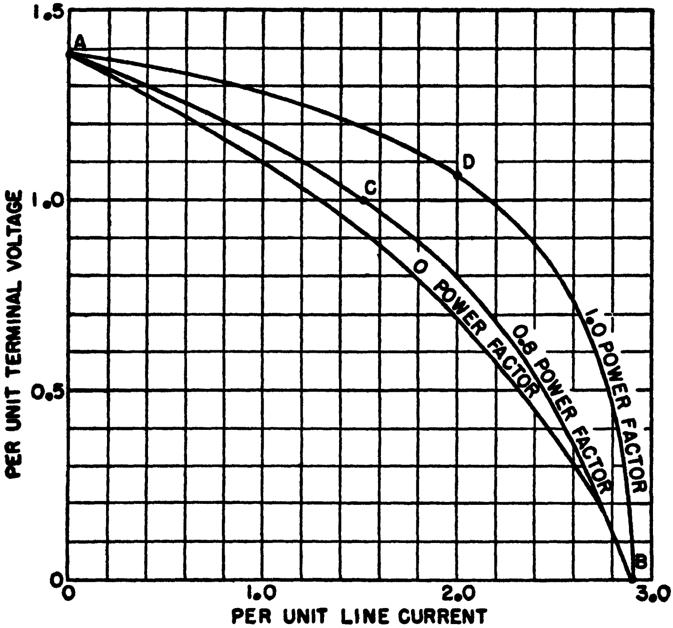

FIG. 13. Volt-ampere characteristics of generator of Fig. 12 at maximum field current.

The field current $I_f = 3.48$ is that required at unit terminal voltage, 150% current, and 0.8 power factor. $V_t = 1.00$, $|I_{a1}| = 1.50$ is point $C$ on the 0.8 power-factor curve. With three points known, a preliminary curve can be drawn. Additional points can be determined by using the preliminary curve for estimating purposes. For example, suppose the preliminary curve passes through the point $V_t = 0.8$, $|I_{a1}| = 2.0$. These values are assumed and $I_f$ calculated. If $I_f < 3.48$, a higher value of $V_t$ is assumed, and $I_f$ again calculated, or estimated.

With two values of $I_f$ which differ but slightly from $I_f = 3.48$, $V_t$ can be satisfactorily estimated. To illustrate: With $|V_t| = 0.8$ and $|I_{a1}| = 2.0$, and a power factor of 0.8,

$$|V_p| = |0.8 + 2.0(0.8 - j0.6)(0.05 + j0.20)| = |1.12 + j0.26| = 1.15$$

$$|E'| = |0.8 + 2.0(0.8 - j0.6)(0.05 + j1.20)| = |2.32 + j1.86| = 2.97$$

at $V_p = 1.15$, $\Delta I_f = 0.50$.

$$I_f = E' + \Delta I_f = 2.97 + 0.50 = 3.47$$

If $|V_t| = 0.81$ and $|I_{a1}| = 2$, $I_f$ will be increased to 3.50. Within the accuracy of plotting the curves the 0.8 power-factor curve passes through the point $|V_t| = 0.8$, $|I_{a1}| = 2.0$, as indicated.

Assume that the unity-power-factor curve passes through the point $|V_t| = 1.0$, $|I_{a1}| = 2.0$:

$$|V_p| = |1.0 + 2.0(0.05 + j0.20)| = |1.10 + j0.40| = 1.17$$

$$|E'| = |1.0 + 2.0(0.05 + j1.20)| = |1.10 + j2.40| = 2.64$$

at $V_p = 1.17$, $\Delta I_f = 0.56$.

$$I_f = 2.64 + 0.56 = 3.20$$

With $V_t = 1\ 1$ and $I_{a1} = 2.0$, $I_f = 3.66$.

By further trials or by estimation, at $I_f = 3.48$ and $|I_{a1}| = 2.0$, $|V_t| = 1.07$, as indicated by point $D$.

Additional points on the curve of Fig. 13 were calculated as illustrated above.

## Calculations of Fundamental-Frequency Currents and Voltages by Means of Symmetrical Components

Although it has been shown that there are two positive-sequence reactances under subtransient, transient, or steady-state conditions of operation, one in the direct and one in the quadrature axis, a synchronous machine can be represented in the positive-sequence network to a sufficiently high degree of precision in many system problems by using only direct-axis reactances or " equivalent " reactances corresponding to subtransient, transient, and sustained conditions which include the effect of saturation.

**Equivalent Circuits To Replace a Synchronous Machine in the Positive-Sequence Network.** In short-circuit calculations involving salient-pole machines, because of the relatively high ratio of reactance to resistance especially in 60-cycle power systems, positive-sequence currents are largely direct-axis components of current which meet direct-axis reactances; the error in assuming them entirely direct-axis components in many cases is small. Or, what is the same in effect, the salient-pole machine can be replaced by a cylindrical-rotor machine in which direct- and quadrature-axes reactances are assumed to be equal. From Figs. 11(*a*) and 11(*b*), it follows that in calculating rms voltage and currents of fundamental frequency, the cylindrical-rotor machine can be replaced in the positive-sequence network by a voltage $E$ behind a positive-sequence reactance $Z_1 = r_1 + jx_1$, as in Fig. 14, where $Z_1$ is the transient, subtransient, or steady-state impedance and $E$ is the voltage behind the impedance. (Figure 14 is the same as Fig. 17, page 43, Volume I.)

The induced current in the additional rotor circuits of a salient-pole machine and in the solid rotor of a cylindrical-rotor machine decays or disappears rapidly, within a few cycles, after a sudden change in armature current. The time required for induced field current to disappear is relatively much longer. (See time constants.) With sudden changes in armature current, subtransient reactance $x_d''$ is used in the study of those phenomena that occur immediately or are of very short duration, such as bus and apparatus stresses, maximum short-circuit currents, maximum fuse and relay currents, and upper limits for circuit-breaker duty. The transient reactance $x_d'$ is used in transient stability calculations and the performance of relays and other devices which depend upon currents which flow after the first few cycles. The synchronous reactance $x_d$ can be used in determining the lower limit of short-circuit current in some types of short-circuit calculations when there are no voltage regulators.

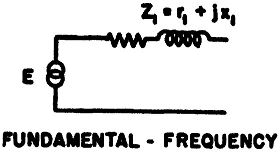

Fig. 14. Fundamental-frequency positive-sequence equivalent circuit of synchronous machine in which reactances in direct and quadrature axes are assumed to be equal.

**Inclusion of Saturation.** The equivalent circuit of Fig. 14 is based on the assumption that the direct- and quadrature-axis reactances, subtransient, transient, or sustained, are equal. Although this assumption may not be true, the effect of the variation of reactances with saturation (neglected in Figs. 10 and 11) is usually greater than the effect of assuming equal reactances in the two axes. When saturated values of reactances corresponding to the degree of saturation in the machine are used in the equivalent circuit of Fig. 14, with the excitation voltage $E$ equal to the equivalent voltage behind its corresponding *saturated reactance*, positive-sequence rms voltages and currents of fundamental frequency throughout the system can be calculated to a closer approximation than with saturation neglected but unequal direct- and quadrature-axis reactances taken into account. An example of this is in steady-state stability calculations, where Fig. 14 is used and the steady-state reactance is adjusted to allow for saturation. As a first approximation,[11] the equivalent reactance of a synchronous generator or motor is taken from 0.6 to 0.8 of $x_d$. In an overexcited synchronous condenser, where the current is almost entirely direct-axis current and the effect of saturation is more pronounced because of the greater amount of iron in the direct than in the quadrature axis, the equivalent reactance is of the order of $0.5x_d$. For a comprehensive treatment of equivalent reactances and the voltages behind these reactances used in steady-state and transient stability calculations see reference 11.

**Equivalent Circuits To Replace a Synchronous Machine in the Negative- and Zero-Sequence Networks.** As negative- and zero-sequence reactances are substantially constant under subtransient, transient, and sustained conditions, the simple equivalent circuits consisting of impedances $Z_2$ and $Z_0$ given in Volume I to represent a synchronous machine in the negative- and zero-sequence networks, respectively, are applicable under all operating conditions where voltages and currents of fundamental frequency only are to be considered.

## TIME CONSTANTS

Any current or voltage, or component of current or voltage, which decreases exponentially with time from an initial value of $I_0$ at time $t = 0$ to a final value of zero, can be expressed as a function of time by the equation

$$i = I_0 \varepsilon^{-t/T} \tag{28}$$

where $T$ is the time constant and is the time in seconds required for the current (or voltage) to decrease to $1/\varepsilon$ or 0.368 of its initial value. If $t = 0$ in the above equation, $i = I_0$; if $t = T$, $i = I_0\varepsilon^{-1} = I_0/\varepsilon$; if $t = \infty$, $i = I_0\varepsilon^{-\infty} = 0$.

In a single series circuit, consisting of resistance $r$ in ohms and inductance $L$ in henries, the time constant $T$ in seconds is $L/r$. If the voltage source is shorted out in this circuit when the current is $I_0$ then, at any subsequent time $t$, $i = I_0\varepsilon^{-t/(L/r)}$. If the time constant $T = L/r$ is large ($r$ small relative to $L$), the current will decrease slowly; but, if $r$ is large relative to $L$, it will decrease rapidly.

In a synchronous machine, because the coupling between armature and rotor circuits differs in the direct and quadrature axes, there are direct-axis and quadrature-axis time constants. For each axis, there are subtransient and transient short-circuit and open-circuit time constants. Subtransient time constants apply to *rapidly decreasing* components and transient time constants to *slowly decreasing* components of current or voltage. There is also the short-circuit time constant of the armature winding which applies to d-c armature current resulting from flux trapped in the phases when short-circuited and to even harmonic and natural-frequency components.

## Open-Circuit Time Constants

*The direct-axis transient open-circuit time constant* $T'_{do}$ is the time in seconds required for the rms a-c value of the *slowly decreasing* component of the quadrature-axis fundamental-frequency armature voltage on open circuit to decrease to $1/\varepsilon = 0.368$ of its value when the field winding is suddenly short-circuited by cutting out the exciter, with the rotor rotating at synchronous speed. As defined here (see Fig. 10), fundamental-frequency open-circuit armature voltage is a quadrature-axis voltage produced by direct-axis field flux.

When the field winding is short-circuited, cutting out the exciter, the exponential decrease of field current and open-circuit armature voltage in a machine without an amortisseur winding depends upon the time constant of the field. If $r_f$ is field resistance in ohms and $L_f$ the field inductance in henries,

$$T'_{do} = \frac{L_f}{r_f} \text{ sec} \qquad [29]$$

This time constant in large machines may be 6 seconds or more. (See Table I.)

In a salient-pole machine with an amortisseur winding, or in a cylindrical-rotor machine, the direct-axis transient open-circuit time constant $T'_{do}$ determines the decrement curve of the rms value of the fundamental-frequency open-circuit armature voltage *shortly after* the field winding is shorted and subtransient effects have disappeared.

The direct-axis subtransient open-circuit time constant, which determines the decrement curve of the *rapidly decreasing* component of rms quadrature-axis fundamental-frequency armature voltage on open circuit when the field winding is suddenly short-circuited, is defined in terms of measurable quantities by considering the armature voltage during the first few cycles after the sudden removal of a three-phase short circuit with the machine running at synchronous speed. Under this condition, there is a negative component of voltage which prevents the armature voltage from rising immediately to the open-circuit value corresponding to impressed field current.

*The direct-axis subtransient open-circuit time constant* $T''_{do}$ is the time in seconds required for the *rapidly decreasing* component of the negative voltage present during the first few cycles in the quadrature-axis component of fundamental-frequency armature voltage under a suddenly removed symmetrical three-phase short circuit, with the machine running at synchronous speed, to decrease to $1/\varepsilon$ or 0.368 of its initial value.

*Quadrature-axis transient and subtransient open-circuit time constants* $T'_{qo}$ and $T''_{qo}$ are analogous to $T'_{do}$ and $T''_{do}$, respectively, the difference being that they are defined in terms of components of voltage in the direct axis instead of in the quadrature axis.

The quadrature-axis transient open-circuit time constant $T'_{qo}$ has no significance in a salient-pole machine, as there is no low-resistance winding in the quadrature axis which corresponds to the field winding in the direct axis. In a cylindrical-rotor machine, after the *rapidly decreasing* induced component of current in the quadrature axis of the machine has disappeared, current continues to flow and produce the same effect that would be produced by a high-resistance short-circuited winding in this axis. $T'_{qo}$ is a much shorter time constant than $T'_{do}$.

## Short-Circuit Time Constants

Consider a three-phase short circuit suddenly applied at the terminals of a salient-pole machine with an amortisseur winding, or a cylindrical-rotor machine, operated at no load with the rotor rotating at synchronous speed and the field winding excited. With excitation in the direct axis only, there will be no quadrature-axis flux at no load; and, neglecting the small resistance of the armature, the fundamental-frequency short-circuit currents will contain only direct-axis components. Oscillograms taken of the currents in the three phases will show a-c components of current whose magnitudes decay rapidly for a few cycles and then decrease slowly until sustained currents result. At least two, and sometimes all three, current waves will be offset initially from their zero axis, in either the positive or the negative direction, indicating the presence of d-c components of current. These d-c currents disappear before the short-circuit current reaches its sustained value.

*The direct-axis transient short-circuit time constant* $T'_d$ is the time in seconds required for the rms value of the *slowly decreasing* component present in the direct-axis component of the a-c component of armature current under suddenly applied symmetrical short-circuit conditions with the rotor rotating at synchronous speed to decrease to $1/\varepsilon$ or 0.368 of its initial value. The *direct-axis subtransient short-circuit time constant* $T''_d$ is similarly defined, except that it applies to the *rapidly decreasing* component instead of the *slowly decreasing* component.

*Quadrature-Axis Short-Circuit Time Constants.* If a synchronous machine is operated under load, there will be flux mutual to armature and rotor in the quadrature axis; therefore, if a three-phase short circuit is suddenly applied under load, there will be quadrature-axis a-c components of current which decrease rapidly in accordance *with the quadrature-axis subtransient short-circuit time constant* $T_q''$, and then less rapidly in accordance with the *quadrature-axis transient short-circuit time constant* $T_q'$. $T_q'$ and $T_q''$ are analogous to $T_d'$ and $T_d''$, respectively, defined above, except that they apply to components of current present in the quadrature-axis component of the a-c component of armature current.

**D-C Components of Current.** When a three-phase short circuit is suddenly applied at the terminals of a synchronous machine, the component of d-c current in any phase depends upon the flux trapped in the phase. As flux and voltage are 90° apart in time phase, the trapped flux and the d-c component of current resulting from it will have their maximum values when the phase voltage is zero. Under this condition, an oscillogram of the current in the phase will show a displaced current wave in which the d-c component $I_{dc}$ is approximately equal to the peak value of the a-c component or $\sqrt{2}$ times its rms value $I$. The effective value of the initial current when the d-c component is a maximum is approximately

$$I_{\text{effective}} = \sqrt{I_{dc}^2 + I^2} = \sqrt{(\sqrt{2}\,I)^2 + I^2} = \sqrt{3}\,I \qquad [30]$$

As the rotor rotates at synchronous speed, the direct and quadrature axes of the rotor alternate rapidly in the positions in line with the axes of the shorted phases. The reactance of the short-circuited armature winding, which depends upon the mutual coupling between armature and rotor circuits, changes rapidly from $x_d''$ to $x_q''$, and is approximately $\frac{1}{2}\,(x_d'' + x_q'') = x_2$.

**The Short-Circuit Time Constant of the Armature Winding $T_a$** is the time in seconds required for the d-c component of armature current under suddenly applied short-circuit conditions, with the rotor rotating at synchronous speed, to decrease to $1/\varepsilon$ or 0.368 of its initial value.

$$T_a = \frac{x_2}{2\pi f r_a} \qquad [31]$$

where $x_2$ is the negative-sequence reactance of the machine and $r_a$ is the d-c resistance of the armature, both expressed in the same unit — ohms or per unit of base impedance.

If there is a symmetrical three-phase impedance $Z_e = r_e + jx_e$ between machine terminals and the short circuit, $T_a$ becomes

$$T_a = \frac{x_2 + x_e}{2\pi f(r_a + r_e)} \qquad [32]$$

**Even-Harmonic Components of Armature Current.** The flux trapped in a short-circuited phase of the armature is not an alternating flux but a flux of constant direction which is decreasing in accordance with the time constant $T_a$. Because of the rotation of the rotor, this flux generates a-c voltages of rated frequency in the direct and quadrature axes of the rotor, which produce currents of rated frequency. An alternating current of rated frequency in the field produces a fundamental space wave of flux which pulsates at rated frequency; this flux wave can be replaced by two fundamental space waves of flux, one rotating forward and one backward at rated speed with respect to the rotor. When referred to the armature, one of these fundamental waves of flux rotates forward at twice synchronous speed, and the other is stationary. The fundamental space wave rotating forward at twice synchronous speed induces a second harmonic voltage in the armature and, as the armature is short-circuited, second harmonic currents will appear in the short-circuit current. These second harmonic currents in turn will produce fourth harmonic armature voltages with resultant fourth harmonic currents in the short-circuited armature. In the general case, there will be all the even-harmonic components of current. However, in most short-circuit calculations, it is unnecessary to consider any even harmonics beyond the zero order — the d-c component of current whose maximum effect on the effective value of initial current is given by [30].

**Relations between Time Constants.** Equivalent circuits to show relations between time constants which determine the decrements of rms fundamental-frequency components of current and voltage in the direct and quadrature axes are shown in Fig. 15, where all quantities are viewed from the armature. The equivalent circuits of Fig. 15 are reciprocal[9] per unit circuits, similar to those of Fig. 9 previously described. The field resistance and leakage reactance are indicated by $r_f$ and $x_{f\ell}$, respectively. It is assumed that amortisseur windings on the salient-pole machine and the analogous effects of the body of the cylindrical-rotor machine can be represented by one resistance and one leakage reactance in each axis: $r_{1d}$ and $x_{1d}$ in the direct axis, and $r_{1q}$ and $x_{1q}$ in the quadrature axis.

In Fig. 15, parts (*a*), (*c*), and (*e*) are similar to parts (*a*), (*b*), and (*c*), respectively, of Fig. 9 except that in (*a*) the resistance $r_f$ of the field is added, and in (*c*) and (*e*) the resistances $r_{1d}$ and $r_{1q}$ of the amortisseur winding in the direct and quadrature axes, respectively.

From Fig. 15(*a*), and [29],

$$T'_{do} = \frac{x_{ad} + x_{f\ell}}{(2\pi f)r_f} = \frac{L_f}{R_f}\,\text{sec} \qquad [33]$$

The open-circuit transient direct-axis time constant $T'_{do} = L_f/R_f$ is readily calculated from $L_f$, the inductance, and $R_f$, the resistance of the field winding.

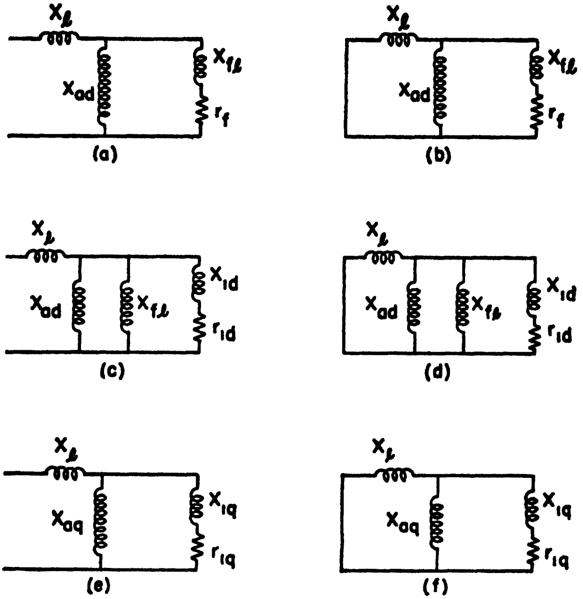

FIG. 15. Reciprocal per unit equivalent circuits for armature and rotor circuits for determining relations between open-circuit and short-circuit time constants.

The direct-axis short-circuit transient time constant $T'_d$ can be expressed in terms of $T'_{do}$, $x_d$, and $x'_d$ by using Fig. 15($b$) and [13], [17], and [33].

$$T'_d = \frac{x_{f\ell} + \dfrac{x_\ell x_{ad}}{x_\ell + x_{ad}}}{(2\pi f) r_f} = T'_{do} \frac{x_\ell(x_{ad} + x_{f\ell}) + x_{ad}x_{f\ell}}{(x_{ad} + x_{f\ell})(x_\ell + x_{ad})} = T'_{do} \frac{x'_d}{x_d} \qquad [34]$$

From Fig. 15($c$),

$$T''_{do} = \frac{x_{1d} + \dfrac{x_{f\ell} x_{ad}}{x_{f\ell} + x_{ad}}}{(2\pi f) r_{1d}} \qquad [35]$$

If the equation for $T''_d$ is written from Fig. 15($d$), this equation can be expressed in terms of $T''_{do}$, $x'_d$, and $x''_d$ by using [17], [18], and [35].

$$T''_d = T''_{do} \frac{x''_d}{x'_d} \qquad [36]$$

From Fig. 15(*e*),

$$T''_{qo} = \frac{x_{1q} + x_{aq}}{(2\pi f)r_{1q}} \qquad [37]$$

From Fig. 15(*f*) and [19], [20], and [37],

$$T''_{q} = T''_{qo}\frac{x''_{q}}{x'_{q}} \qquad [38]$$

By analogy from [34], $T'_q$ can be expressed in terms of $T'_{qo}$, $x'_q$, and $x_q$ for a cylindrical-rotor machine in which $T'_{qo}$ is appreciable.

$$T'_{q} = T'_{qo}\frac{x'_{q}}{x_{q}} \qquad [39]$$

**External Reactance.** If there is a symmetrical three-phase reactance $x_e$ between machine terminals and the short circuit, $x_e$ is part of the armature circuit and is added to $x_d$, $x'_d$, $x''_d$, $x_q$, $x'_q$, and $x''_q$ in the above equations, giving

$$T'_{d} = T'_{do}\frac{x'_{d} + x_e}{x_d + x_e} \qquad [40]$$

$$T''_{d} = T''_{do}\frac{x''_{d} + x_e}{x'_d + x_e} \qquad [41]$$

$$T''_{q} = T''_{qo}\frac{x''_{q} + x_e}{x'_q + x_e} \qquad [42]$$

$$T'_{q} = T'_{qo}\frac{x'_{q} + x_e}{x_q + x_e} \qquad [43]$$

*The fundamental-frequency rms direct-axis short-circuit current at any time during a symmetrical three-phase fault* at the machine terminals, from a condition of no-load and rated voltage, is given by the equation

$$i = (i'' - i')\varepsilon^{-t/T''_d} + (i' - I)e^{-t/T'_d} + I \qquad [44]$$

where the rated per unit no-load voltage is unity, and

$i$ = per unit rms fundamental-frequency current at any time $t$ in seconds

$i'' = \frac{1}{x''_d}$ = initial rms fundamental-frequency current.

$i' = \frac{1}{x'_d}$ = initial rms fundamental-frequency current with high-decrement component of current $(i'' - i')$ neglected.

$I = \dfrac{1}{x_d}$ = sustained rms fundamental-frequency current

$T_d''$ and $T_d'$ = subtransient and transient direct-axis short-circuit time constants, respectively.

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# CHAPTER VIII

# $\alpha\beta 0$ COMPONENTS IN SYNCHRONOUS-MACHINE ANALYSIS

In the general analysis of a three-phase system, three sets of components are required. Because the synchronous machine has two axes of rotor symmetry, the components generally best suited to its analysis are direct-axis, quadrature-axis, and zero-sequence components. In system studies where fundamental-frequency currents and voltages only need be considered, the method of symmetrical components is well adapted to the solution of problems involving rotating machines under both balanced and unbalanced conditions. This has been illustrated in Volume I and discussed in the preceding chapter. During unbalanced conditions where phase currents and voltages may include appreciable harmonic, natural-frequency and d-c components as well as fundamental-frequency terms, $\alpha\beta 0$ components (Volume I, Chapter X) provide a natural transition from phase quantities to direct-axis, quadrature-axis, and zero-sequence components. Zero-sequence components (0 components) are exactly the same in both systems.

Equations [13]–[18] of Chapter I summarize the relations between phase voltages and their $\alpha\beta 0$ components. Equations of the same form relate instantaneous phase voltages, currents, and flux linkages and their instantaneous $\alpha\beta 0$ components.

**Assumptions.** Saturation and hysteresis in the magnetic circuits and eddy current in the iron will be neglected, and an *ideal synchronous machine* will be assumed. An ideal synchronous machine has been defined as one having the same self- and mutual armature leakage reactances as the actual machine which it replaces " but ideal to the extent that, as far as concern effects depending on the position of the rotor, each armature winding is sinusoidally distributed."[1,2]

Under the assumption of a sinusoidally distributed armature winding, armature currents produce only fundamental space waves of mmf and harmonic space waves of flux rotating with respect to the armature produce no resultant armature voltages. It was pointed out in the preceding chapter that armature voltages resulting from rotating harmonic space waves of flux are of small magnitude relative to the fundamental in a modern synchronous machine, as they are reduced by the order of the harmonic and by pitch and distribution factors; in the ideal synchronous machine, they disappear entirely. This applies to all harmonic space waves of flux, regardless of their speed of rotation. However, all rotating *fundamental* space waves of flux produce armature voltages of the same harmonic frequency as the speed of the rotating fundamental flux wave relative to synchronous speed. For example, a fundamental space wave of flux rotating forward relative to the armature at three times synchronous speed will produce positive-sequence third harmonic armature voltages.

**Per Unit Armature Quantities.** The per unit quantities used here are defined as follows. Unit armature current and voltage are crest values of rated phase current and rated phase voltage, respectively; unit impedance is the ratio of rated phase voltage to rated phase current; unit speed of the rotor is synchronous speed; unit time is one electrical radian; unit armature linkage is that linkage the time rate of change of which will generate unit armature voltage; unit armature mmf is the mmf produced by rated positive-sequence armature currents; unit permeance is that permeance which when multiplied by unit mmf produces unit armature linkages.

**Notation.** Let $i$, $\psi$, and $e$ with appropriate subscripts represent instantaneous per unit armature current, flux linkage, and voltage, respectively; let subscripts $a$, $b$, $c$ refer to phases $a$, $b$, $c$, respectively; let subscripts $\alpha$, $\beta$, 0 refer to $\alpha$, $\beta$, and 0 (zero-sequence) components, respectively; let subscripts $d$ and $q$ refer to direct- and quadrature-axis components, respectively.

**Conventions for Signs.** The conventions for signs used here are those of references 1 and 2. In the equivalent two-pole synchronous machine of Fig. 1, the axes of the phases, the direct and quadrature axes of the poles, and the direction of rotation of the rotor are indicated. The $\alpha$ axis coincides with the axis of phase $a$. The direct axis lines up with the axes of the phases in the order $a$, $b$, $c$; it lines up with the $\alpha$ and $\beta$ axes in the order $\alpha$, $\beta$, so that at no load with the rotor rotating at synchronous speed, generated $e_\alpha$ leads generated $e_\beta$ by 90°.

Directions of magnetization of positive $i_d$ and $i_q$, defined in Chapter VII, are fixed relative to the rotor. Directions of magnetization of positive phase currents $i_a$, $i_b$, and $i_c$ and of positive $i_\alpha$ and $i_\beta$ are fixed relative to the armature.

In an armature circuit having resistance $r$, current $i$, and flux linkage $\psi_i$ caused by current $i$ *only*, the terminal voltage $e_t$ at any instant may be written

$$e_t = -p\psi_i - ri$$

where $p = d/dt$ = rate of change with respect to time, and positive direction for $i$ is toward the terminal. If part of the flux linking the circuit is due to the mutual coupling with another circuit (the field, for example), and this mutual flux $\psi_f$ *opposes* $\psi_i$, the equation for $e_t$ becomes

$$e_t = p(\psi_f - \psi_i) - ri = p\psi - ri$$

where $\psi = \psi_f - \psi_i$ = total flux linking the circuit, and the linkage $\psi_i$ due to current $i$ in the circuit in the positive direction is written with a negative sign. The mmf producing $\psi_i$ is likewise written with a negative sign.

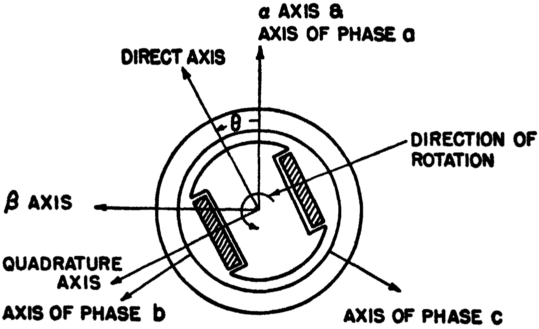

FIG. 1. Position of direct and quadrature axes of the rotor, for a given value of $\theta$, relative to the $\alpha$ and $\beta$ axes of the armature.

In accordance with this convention, the terminal voltages of the $\alpha$, $\beta$, and 0 circuits of a synchronous machine will be written

$$e_\alpha = p\psi_\alpha - ri_\alpha \quad [1]$$

$$e_\beta = p\psi_\beta - ri_\beta \quad [2]$$

$$e_0 = p\psi_0 - ri_0 \quad [3]$$

where $\psi_\alpha$, $\psi_\beta$, and $\psi_0$ are total instantaneous per unit flux linkages with the $\alpha$, $\beta$, 0 circuits, respectively; $r$ is the armature resistance, assumed the same in all three circuits; and the direction of $i_\alpha$, $i_\beta$, and $i_0$ is toward the terminals. $\psi_\alpha$ and $\psi_\beta$ are per unit armature flux linkages in $\alpha$ and $\beta$ circuits due to currents in both armature and rotor circuits. Per unit $\psi_0$, which is independent of rotor currents, is

$$\psi_0 = -i_0x_0 \quad [4]$$

where $x_0$, the per unit zero-sequence fundamental-frequency reactance, is equal numerically to per unit zero-sequence inductance.

**Relations between Instantaneous Phase Quantities and their αβ0 Components.** At any instant,

(a) $$i_a = i_\alpha + i_0$$

(b) $$i_b = -\frac{1}{2}i_\alpha + \frac{\sqrt{3}}{2}i_\beta + i_0$$

(c) $$i_c = -\frac{1}{2}i_\alpha - \frac{\sqrt{3}}{2}i_\beta + i_0$$

(d) $$i_\alpha = \frac{2}{3}\left(i_a - \frac{i_b + i_c}{2}\right) \qquad [5]$$

(e) $$i_\beta = \frac{1}{\sqrt{3}}(i_b - i_c)$$

(f) $$i_0 = \frac{1}{3}(i_a + i_b + i_c)$$

If $i$ in the above equations is replaced by $\psi$ or $e$, equations relating instantaneous flux linkages or voltages, respectively, in the phases and in the $\alpha$, $\beta$, and 0 circuits will be obtained.

**Relations between α and β Components and Direct- and Quadrature-Axis Components.** Instantaneous $i_d$ and $i_q$ produce instantaneous fundamental mmf space waves which have per unit crests in the direct and quadrature axes equal to per unit $i_d$ and $i_q$, respectively. Likewise, instantaneous $i_\alpha$ and $i_\beta$ produce instantaneous fundamental mmf space waves which have per unit crests in the $\alpha$ and $\beta$ axes equal to per unit $i_\alpha$ and $i_\beta$, respectively. With all quantities in per unit, and the direct axis displaced from the axis of phase $a$ in the normal direction of rotation of the rotor by an angle $\theta$, $i_d$ and $i_q$ are expressed in terms of $i_\alpha$ and $i_\beta$, and $i_\alpha$ and $i_\beta$ in terms of $i_d$ and $i_q$, by simple projections in Fig. 1:

$$i_d = i_\alpha \cos\theta + i_\beta \sin\theta \qquad [6]$$

$$i_q = -i_\alpha \sin\theta + i_\beta \cos\theta \qquad [7]$$

$$i_\alpha = i_d \cos\theta - i_q \sin\theta \qquad [8]$$

$$i_\beta = i_d \sin\theta + i_q \cos\theta \qquad [9]$$

If $i$ in the above equations is replaced by $\psi$ or $e$, equations relating direct- and quadrature-axis components and $\alpha$ and $\beta$ components of flux linkages or voltages, respectively, will be obtained:

$$\psi_d = \psi_\alpha \cos\theta + \psi_\beta \sin\theta \qquad [10]$$

$$\psi_q = -\psi_\alpha \sin\theta + \psi_\beta \cos\theta \qquad [11]$$

$$e_d = e_\alpha \cos\theta + e_\beta \sin\theta \qquad [12]$$

$$e_q = -e_\alpha \sin\theta + e_\beta \cos\theta \qquad [13]$$

$$\psi_\alpha = \psi_d \cos\theta - \psi_q \sin\theta \qquad [14]$$

$$\psi_\beta = \psi_d \sin\theta + \psi_q \cos\theta \qquad [15]$$

$$e_\alpha = e_d \cos\theta - e_q \sin\theta \qquad [16]$$

$$e_\beta = e_d \sin\theta + e_q \cos\theta \qquad [17]$$

**Park's Equations for an Ideal Synchronous Machine.** If $e_\alpha$ and $e_\beta$ from [1] and [2] are substituted in [12] and [13],

$$\begin{aligned} e_d &= \cos\theta(p\psi_\alpha - ri_\alpha) + \sin\theta(p\psi_\beta - ri_\beta) \\ &= \cos\theta p\psi_\alpha + \sin\theta p\psi_\beta - r(i_\alpha \cos\theta + i_\beta \sin\theta) \qquad [18] \\ &= \cos\theta p\psi_\alpha + \sin\theta p\psi_\beta - ri_d \end{aligned}$$

$$\begin{aligned} e_q &= -\sin\theta(p\psi_\alpha - ri_\alpha) + \cos\theta(p\psi_\beta - ri_\beta) \\ &= -\sin\theta p\psi_\alpha + \cos\theta p\psi_\beta - ri_q \qquad [19] \end{aligned}$$

By differentiation of [10] and substitutions from [18] and [11],

$$\begin{aligned} p\psi_d &= p(\psi_\alpha \cos\theta + \psi_\beta \sin\theta) \\ &= (\cos\theta p\psi_\alpha + \sin\theta p\psi_\beta) + (-\psi_\alpha \sin\theta + \psi_\beta \cos\theta)p\theta \\ &= e_d + ri_d + \psi_q p\theta \qquad [20] \end{aligned}$$

By differentiation of [11] and substitutions from [19] and [10],

$$\begin{aligned} p\psi_q &= p(-\psi_\alpha \sin\theta + \psi_\beta \cos\theta) \\ &= (-\sin\theta p\psi_\alpha + \cos\theta p\psi_\beta) - (\psi_\alpha \cos\theta + \psi_\beta \sin\theta)p\theta \\ &= e_q + ri_q - \psi_d p\theta \qquad [21] \end{aligned}$$

From [20] and [21],

$$e_d = p\psi_d - ri_d - \psi_q p\theta \qquad [22]$$

$$e_q = p\psi_q - ri_q + \psi_d p\theta \qquad [23]$$

There is also from [3] and [4], since $x_0$ is substantially constant under all operating conditions,

$$e_0 = -(r_0 + x_0 p)i_0 \qquad [24]$$

Equations [22]–[24] are the well-known Park equations. The development of [22] and [23] given above follows that of reference 2, except that $\alpha\beta 0$ components are used instead of phase quantities. In order to apply [22] and [23], equations for the per unit armature flux linkages $\psi_d$ and $\psi_q$ in the direct and quadrature axes, respectively, and $p\theta$ the speed of the rotor in per unit of synchronous speed are required. Equations for $\psi_d$ and $\psi_q$ which include the time constants of the various circuits involved, and the change in the voltage of the field exciter due to action of voltage regulators, are given in references 2 and 3 and elsewhere in the literature. A study involving change in rotor speed is given in reference 4.

Because of the differential equations involved, many synchronous machine problems are most simply solved by means of a differential analyzer.[5,6] Examples of solutions by means of a differential analyzer are found in references 4 and 7. In such solutions, smooth curves of voltages and currents under specified conditions can be very quickly obtained. In analytic solutions, even in the simplest cases, it is necessary to make simplifying assumptions or to use laborious time-consuming step-by-step methods.

In this chapter only those problems will be discussed which can be solved by assuming constant rotor speed equal to synchronous speed, and constant voltage of the field exciter.

At synchronous speed, the speed of the rotor in per unit of synchronous speed is

$$p\theta = \frac{d\theta}{dt} = 1$$

where $\theta$ is the electrical angular displacement in radians at time $t$ in electrical radians of the direct axis of the rotor from the axis of phase $a$, measured in the normal direction of rotation of the rotor. At any time $t$,

$$\theta = \theta_0 + t \qquad [25]$$

where $\theta_0 = \theta$ at time $t = 0$ from which time is measured.

At constant voltage of the field exciter, let the per unit d-c field current resulting from exciter voltage be indicated by $I_f$, where

$I_f$ = d-c field current in per unit of that field current which produces a unit space fundamental flux wave in the air gap and generates unit armature voltages at no load with saturation neglected.

Under the assumptions of constant rotor speed and constant exciter voltage, simple equations can be written for $\psi_d$ and $\psi_q$ which apply during normal balanced operation and at the first instant during a disturbance on a previously balanced system.

The per unit flux linkages $\psi_d$ and $\psi_q$ with the armature winding in the direct and quadrature axes, respectively, *due to the per unit d-c field current* $I_f$ are $\psi_d = I_f$ and $\psi_q = 0$. On these flux linkages with the armature winding must be superposed those *due to the components of armature current* $i_d$ and $i_q$ in the direct and quadrature axes, respectively. The per unit crests of the fundamental mmf space waves in the direct and quadrature axes produced by $i_d$ and $i_q$ are numerically equal to $i_d$ and $i_q$, respectively. Per unit fundamental-frequency reactances are numerically equal to corresponding per unit inductances. Per unit armature flux linkages produced by armature currents $i_d$ and $i_q$ are therefore conveniently expressed in terms of per unit direct- and quadrature-axes reactances.

The factors of proportionality (per unit permeances) between fundamental space wave of armature mmf's and flux linkages in the direct and quadrature axes for suddenly applied $i_d$ and $i_q$ are $x_d''$ and $x_q''$, respectively; they are $x_d'$ and $x_q'$ for suddenly applied $i_d$ and $i_q$ in a machine without an amortisseur winding or its equivalent; under sustained balanced operation they are $x_d$ and $x_q$. In accordance with the convention chosen for signs, if the mmf due to $i_d$ opposes the mmf of the main field winding, $i_d$ is positive but the mmf due to $i_d$ and the resultant flux linkages are written with negative signs. This is explained in the development of [1]–[4]. The mmf and flux linkages with the armature due to positive $i_q$ are likewise written with negative signs. Positive $i_q$ leads positive $i_d$.

**Normal Balanced Operation.** Under sustained balanced operation in a symmetrical system, currents and voltages are of positive sequence. Per unit linkage $\psi_d$ and $\psi_q$ due to positive-sequence currents replaced by their direct- and quadrature-axis components $i_d$ and $i_q$, respectively, are $-i_d x_d$ and $-i_q x_q$. The total per unit flux linkage $\psi_d$ and $\psi_q$ with the armature winding in the direct and quadrature axes, respectively, are

$$\begin{aligned} \psi_d &= I_f - i_d x_d \\ \psi_q &= -i_q x_q \end{aligned} \qquad [26]$$

As $I_f$, $i_d$, and $i_q$ are constant for sustained balanced operation, $p\psi_d = p\psi_q = 0$. Equations [22] and [23] become, with $p\theta = 1$,

$$\begin{aligned} e_d &= -ri_d + x_q i_q \\ e_q &= I_f - ri_q - x_d i_d \end{aligned} \qquad [27]$$

From [27], the positive-sequence voltages and currents of the reference phase $a$ can be determined throughout the system as illustrated in the preceding chapter; and from them the $\alpha$ and $\beta$ components. Under balanced operations in a symmetrical system, there are no negative- and zero-sequence components. From [5], $e_\alpha = e_a$ and $i_\alpha = i_a$; $e_\beta$ and $I_\beta$ are equal in magnitude to $e_\alpha$ and $i_\alpha$, respectively, but lag them by 90° in time phase.

$\alpha$ and $\beta$ components are not well adapted to the solution of normal balanced systems; but in the study of unbalanced conditions during disturbances, the $\alpha$ and $\beta$ voltages or currents just previous to the occurrence of the disturbance will be required. Under balanced load, in a symmetrical system, let instantaneous $e_\alpha$ and $e_\beta$ at any system point $P$ be written

$$\begin{aligned} e_\alpha &= e_a = -V_p \sin(\theta + \phi) \\ e_\beta &= -je_\alpha = -V_p \sin(\theta + \phi - 90°) = V_p \cos(\theta + \phi) \end{aligned} \qquad [28]$$

where $V_p$ is the per unit crest of the fundamental-frequency voltage at $P$, $\phi$ is the load angle, and $\theta$ the angular displacement of the direct axis from the axis of phase $a$.

**No Load.** At no load and synchronous speed, from [26],

$$\psi_d = I_f; \quad \psi_q = 0$$

from [14] and [15],

$$\begin{aligned} \psi_\alpha &= \psi_d \cos\theta = I_f \cos\theta \\ \psi_\beta &= \psi_d \sin\theta = I_f \sin\theta \end{aligned} \qquad [29]$$

Substitution of [29] in [1] and [2] gives the no-load $\alpha$ and $\beta$ voltages:

$$\begin{aligned} e_\alpha &= -I_f \sin\theta \\ e_\beta &= I_f \cos\theta \end{aligned} \qquad [30]$$

From [5], the no-load phase voltages are

$$\begin{aligned} e_a &= e_\alpha = -I_f \sin\theta \\ e_b &= -\frac{1}{2}e_\alpha + \frac{\sqrt{3}}{2}e_\beta = \frac{1}{2}I_f \sin\theta + \frac{\sqrt{3}}{2}I_f \cos\theta \\ e_c &= -\frac{1}{2}e_\alpha - \frac{\sqrt{3}}{2}e_\beta = \frac{1}{2}I_f \sin\theta - \frac{\sqrt{3}}{2}I_f \cos\theta \end{aligned} \qquad [31]$$

**Disturbances.** A disturbance is any sudden change in system operating conditions. It includes such conditions as short circuits, the opening of a conductor or conductors, and increase or decrease in load.

**Superposition.** In a system in which resistance, inductance, capacitance, and leakance are constant, voltages and currents at various system points during a disturbance can be determined by superposing the components of current and voltage resulting from the disturbance upon the voltages and currents which would exist at these points if the disturbance had not occurred. Components of voltages and currents at various system points *due to a disturbance*, such as an unsymmetrical short circuit which reduces a phase voltage or components of phase voltage to zero, can be determined by applying a voltage or components of voltage at the point of fault equal in magnitude but opposite in sign to the voltage or components of voltage which existed there at the instant the fault occurred. Similarly, if a phase current or components of phase current are reduced to zero by the disturbance, such as the opening of a conductor, the components of voltage and current at various system points due to the disturbance can be determined by applying a current or components of current across the opening equal in magnitude but opposite in sign to the current or components of current which existed there at the instant the disturbance occurred.

**Conditions during Disturbances.** Action of voltage regulators and change in rotor speed do not occur instantly. Therefore at the *first instant* during a disturbance, the voltage of the field exciter and the speed of the rotor remain constant. As explained in the preceding chapter, flux linkages in the closed circuits of amortisseur and field cannot change instantly. The mmf's resulting from currents induced in them oppose the mmf's due to suddenly applied armature currents, so that the armature flux is forced into leakage paths of low permeance. The per unit mmf's due to per unit $\Delta i_d$ and $\Delta i_q$ in the direct and quadrature axes are $-\Delta i_d$ and $-\Delta i_q$, respectively. As previously stated, the factors of proportionality between armature flux and mmf in these axes for suddenly applied currents are $x_d''$ and $x_q''$. Armature flux linkages in the direct and quadrature axes are therefore decreased by $\Delta i_d x_d''$ and $\Delta i_q x_q''$. In a salient-pole machine without an amortisseur winding, they are decreased by $\Delta i_q x_d'$ and $\Delta i_q x_q' = \Delta i_q x_q$. As the induced rotor currents decay in amortisseur and field in accordance with their time constants, armature flux enters paths of higher permeance, and flux linkages in the armature circuits are further reduced. Since time constants of amortisseur windings are small, currents induced in them are of short duration. Since the time constant of the field is large, induced field currents decay slowly. Figure 2 (taken from reference 8) shows the relative attenuating effect of these two rotor time constants on direct-axis fundamental-frequency armature current during a three-phase short circuit.

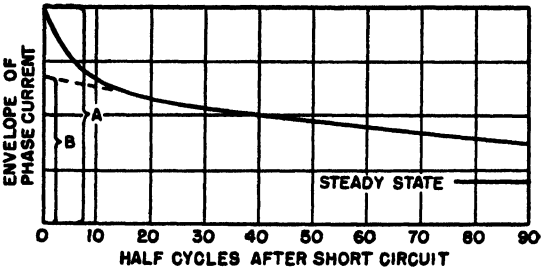

FIG. 2. Determination of $x_d'$ and $x_d''$ from test oscillogram. Curve shows envelope of symmetrical short-circuit phase-current wave. $x_d''$ inversely proportional to initial current $A$. Dotted curve shows envelope of symmetrical current after high-speed transient has been subtracted. $x_d'$ inversely proportional to $B$.

**Initial Conditions.** At the first instant, the changes $\Delta\psi_d$ and $\Delta\psi_q$ in the armature flux linkages due to sudden positive increments $\Delta i_d$ and $\Delta i_q$ in the direct- and quadrature-components of armature currents are

$$\Delta\psi_d = -x_d''\Delta i_d = -x_d''(\Delta i_\alpha \cos\theta + \Delta i_\beta \sin\theta) \qquad [32]$$

$$\Delta\psi_q = -x_q''\Delta i_q = -x_q''(-\Delta i_\alpha \sin\theta + \Delta i_\beta \cos\theta) \qquad [33]$$

where $\Delta i_d$ and $\Delta i_q$ have been replaced by their values from [6] and [7] in terms of $\Delta i_\alpha$ and $\Delta i_\beta$.

If $\Delta\psi_d$ and $\Delta\psi_q$ from [32] and [33] are substituted in [14] and [15], $\Delta\psi_\alpha$ and $\Delta\psi_\beta$ in terms of $\Delta i_\alpha$ and $\Delta i_\beta$ are obtained:

$$\Delta\psi_\alpha = -\Delta i_\alpha(x_d'' \cos^2\theta + x_q'' \sin^2\theta) - \Delta i_\beta(x_d'' - x_q'')\sin\theta\cos\theta \quad [34]$$

$$\Delta\psi_\beta = -\Delta i_\alpha(x_d'' - x_q'')\sin\theta\cos\theta - \Delta i_\beta(x_d'' \sin^2\theta + x_q'' \cos^2\theta) \quad [35]$$

Let $x = \frac{1}{2}(x_d'' + x_q'')$ and $y = \frac{1}{2}(x_d'' - x_q'')$; then, $x_d'' = x + y$ and $x_q'' = x - y$. If these substitutions are made in the above equations, $\cos^2\theta - \sin^2\theta$ replaced by $\cos 2\theta$, and $\sin\theta\cos\theta$ by $\frac{1}{2}\sin 2\theta$, there results

$$\Delta\psi_\alpha = -\Delta i_\alpha(x + y\cos 2\theta) - \Delta i_\beta\, y \sin 2\theta \qquad [36]$$

$$\Delta\psi_\beta = -\Delta i_\alpha\, y\sin 2\theta - \Delta i_\beta(x - y\cos 2\theta) \qquad [37]$$

The changes $\Delta e_\alpha$ and $\Delta e_\beta$ in the $\alpha$ and $\beta$ terminal voltages for sudden changes $\Delta i_\alpha$ and $\Delta i_\beta$ in $\alpha$ and $\beta$ currents in the machine flowing toward its terminals are obtained by substitution of [36] and [37] in [1] and [2]:

$$\Delta e_\alpha = -[r + p(x + y\cos 2\theta)]\Delta i_\alpha - [py\sin 2\theta]\Delta i_\beta \qquad [38]$$

$$\Delta e_\beta = -[py\sin 2\theta]\Delta i_\alpha - [r + p(x - y\cos 2\theta]\Delta i_\beta \qquad [39]$$

where

$$x = \tfrac{1}{2}(x_d'' + x_q'') \quad \text{and} \quad y = \tfrac{1}{2}(x_d'' - x_q'') \qquad [40]$$

In a salient-pole machine without an amortisseur winding or its equivalent, $x_d'' = x_d'$ and $x_q'' = x_q' = x_q$.

Equations [38]–[40] give initial relations between increments of $\alpha$ and $\beta$ components of voltage and current at the machine terminals for a sudden change in any of these components. These equations, developed for initial conditions, are applicable at subsequent times if appropriate time constants are applied. They will be applied here during the first half cycle following an unsymmetrical short circuit to determine approximate overvoltages resulting from the short circuit. They have been used to determine circuit breaker recovery voltages[9] and harmonic currents and voltages resulting from unsymmetrical short circuits.[10]

## UNBALANCED SHORT CIRCUITS

If armature resistance (as well as rotor resistances) is neglected, [38]–[40] together with [24] can be used to determine the voltages which occur during the first half cycle following suddenly applied unsymmetrical short circuits. If $x''$ and $y''$ defined in [40] and $x_0$ in [24] include external reactance in series with the machine, transformer and line reactances up to point of fault will be included. By neglecting armature resistances the attenuation of armature transients is neglected, but their initial amplitudes are not appreciably affected.

### Machine Unloaded, Resistance and Capacitance Neglected

Let the machine be operated at rated voltage and synchronous speed.

**Line-to-Line Short Circuit between Phases $b$ and $c$.** The fault conditions are: $e_b = e_c$; $i_b = -i_c$; $i_a = 0$. From [5] and similar voltage equations: $i_0 = 0$; $i_\alpha = 0$; $e_\beta = 0$. At no load and rated voltage, $I_f = 1$, and from [30], just previous to the instant of short circuit,

$$e_\beta = I_f\cos\theta = \cos\theta = \cos(\theta_0 + t) \qquad [41]$$

If $\Delta e_\beta$ in [39] is replaced by $-\cos\theta$ (the negative of the voltage $e_\beta$ given by [41]), $\Delta i_\alpha$ by zero, and armature resistance neglected,

$$p(x - y\cos 2\theta)i_\beta = \cos\theta = \cos(\theta_0 + t)$$

Integrating both sides of the above equation and solving for $i_\beta$ gives

$$i_\beta = \frac{\sin\theta - \sin\theta_0}{x - y\cos 2\theta}$$

From [38] with $i_\alpha = 0$,

$$\Delta e_\alpha = -p\,\frac{y\sin 2\theta\,(\sin\theta - \sin\theta_0)}{x - y\cos 2\theta}$$

The voltage $e_a$ of the unfaulted phase $= e_\alpha$, which is the sum of $e_\alpha$ before the fault (given by [30] with $I_f = 1$) and $\Delta e_\alpha$ given above.

$$e_a = e_\alpha = -\sin\theta - p\,\frac{y\sin 2\theta\,(\sin\theta - \sin\theta_0)}{x - y\cos 2\theta}$$

$$= -\sin\theta - \frac{\left.\begin{array}{l}(x - y\cos 2\theta)\,[2y\cos 2\theta\,(\sin\theta - \sin\theta_0)\\ \quad + y\sin 2\theta\cos\theta] - 2y^2\sin^2 2\theta\,(\sin\theta - \sin\theta_0)\end{array}\right\}}{(x - y\cos 2\theta)^2}$$

$$= -\sin\theta - \frac{2y}{(x - y\cos 2\theta)^2}\,[x\sin\theta(\cos 2\theta + \cos^2\theta)$$
$$- y\sin\theta(1 + \cos 2\theta\cos^2\theta) - \sin\theta_0\,(x\cos 2\theta - y)] \qquad [42]$$

The value $\theta_0$ of $\theta$ at time $t = 0$ determines the flux linkages trapped in the $\beta$ circuit. From [29], $\psi_\beta$ has its maximum value at $t = 0$ when $\theta_0 = \pm\pi/2$ and its minimum value when $\theta_0 = 0$ or $\pi$.

*Maximum Flux Linkages at* $t = 0$. If $\theta_0 = \pi/2$ is substituted in [42],

$$e_a = -\sin\theta - \frac{2y}{(x - y\cos 2\theta)^2}\,\{\sin\theta\,[x\,(\cos 2\theta + \cos^2\theta)$$
$$- y\,(1 + \cos 2\theta\cos^2\theta)] - (x\cos 2\theta - y)\} \qquad [43]$$

where $\theta = \theta_0 + t = (\pi/2) + t$.

A graph of $e_a$ versus time can be plotted for [43] which will show maximum and minimum values of $e_a$. Or, more simply, if [43] is differentiated with respect to $\theta$ and $de_a/d\theta$ equated to zero, the value of $\theta$ corresponding to maximum and minimum $e_a$ can be obtained. If this differentiation is performed, it will be seen that $\cos\theta$ is a factor of $de_a/d\theta$; therefore $e_a$ has a maximum or minimum value if $\cos\theta = 0$, or $\theta = \pi/2$ or $\frac{3}{2}\pi$.

If $\theta = \theta_0 + t = \pi/2 + t = \pi/2,\ t = 0$ and $e_a$ in [42] is

$$e_a = -1 - \frac{2y}{(x + y)^2}\,[-x - y - (-x - y)] = -1$$

If $\theta = \theta_0 + t = \pi/2 + t = \frac{3}{2}\pi$, $t = \pi$. With $\theta$ in [42] replaced by $\frac{3}{2}\pi$, $e_a$ is

$$e_a = 1 - \frac{2y}{(x+y)^2}(2x + 2y) = 1 - \frac{4y}{x+y}$$

Substitution of $x$ and $y$ from [40] in the above equation gives

$$e_a = 1 - \frac{2(x_d'' - x_q'')}{x_d''} = \frac{2x_q''}{x_d''} - 1 \qquad [44]$$

Equation [44] gives the maximum value of $e_a$ at maximum flux linkages at $t = 0$. This value occurs at $t = \pi$ or one-half cycle after the instant of fault.

*Minimum Flux Linkages at* $t = 0$. If $\theta_0 = 0$ is substituted in [42], the term $\sin\theta_0(x\cos 2\theta - y)$ disappears. If $\theta = t = \pi/2$ is then substituted:

$$e_a = -1 + \frac{2y(x+y)}{(x+y)^2} = -1 + \frac{2y}{x+y} = \frac{y-x}{y+x} = -\frac{x_q''}{x_d''} \qquad [45]$$

Equation [45] gives the maximum value of $e_a$ with minimum flux linkages at $t = 0$. This value occurs at $t = \pi/2$ or one-quarter cycle after the instant of fault.

Equations [44] and [45] agree with those given by Doherty and Nickle.[11]

**Problem 1.** In a salient-pole machine without an amortisseur winding, $x_d'' = x_d' = 0.30$ and $x_q'' = x_q' = x_q = 0.70$ in per unit based on the machine rating. A line-to-line fault occurs at the terminals of the unloaded machine operated at rated speed and voltage. What is the peak value of the voltage of the unfaulted phase in per unit of normal peak voltage?

*Solution.* The maximum voltage at maximum flux linkages from [44] is

$$e_a = \frac{2 \times 0.70}{0.30} - 1 = 3.667 \text{ times normal peak voltage}$$

The maximum voltage at minimum flux linkages from [45] is

$$e_a = -\frac{0.70}{0.30} = -2.333 \text{ times normal peak voltage}$$

As the fault may occur at maximum or minimum flux linkages or at any intermediate value, the peak voltage of the unfaulted phase will be between 2.33 and 3.67 times normal, neglecting decrements, and will occur within the first half cycle.

**Even and Odd Harmonics.** With flux trapped in the $\beta$ circuit there will be even harmonics including a d-c component and odd harmonics including a fundamental-frequency term. All harmonics are included in [43]. Only odd harmonics are present when the fault occurs at zero flux linkages in the $\beta$ circuit. Even harmonics are attenuated by the armature time constant; odd harmonics by the amortisseur and field time constants. (See Time Constants, Chapter VII.) In the above problem there is no amortisseur winding and, as the field time constant is large, the calculated voltage $e_a$ at minimum flux linkages which occurs one-quarter cycle after the instant of fault is substantially correct. The value at maximum flux linkages, however, is too high because of neglecting the effect of the armature time constant in reducing even-harmonic components.

**Line-to-Ground Short Circuit on Phase *a*.** The fault conditions are $e_a = 0$; $i_b = 0$; $i_c = 0$. From [5],

$$i_\beta = 0 \qquad i_\alpha = \tfrac{2}{3}i_a = 2i_0 \qquad e_0 = -e_\alpha \tag{46}$$

From [24], neglecting resistance, $e_0 = -x_0 p i_0$. After the fault,

$$e_\alpha = -e_0 = x_0 p i_0 = \tfrac{1}{2}x_0 p i_\alpha$$

From [30], before the fault, $e_\alpha = -I_f \sin\theta = -\sin\theta$.

$$\begin{aligned}\Delta e_\alpha &= \text{the change in voltage because of fault}\\ &= -(-\sin\theta) + \tfrac{1}{2}x_0 p i_\alpha.\end{aligned}$$

If $\Delta e_\alpha$ is replaced by the above value and $i_\beta$ by zero, [38] with $r$ neglected becomes

$$\sin\theta + \tfrac{1}{2}x_0 p i_\alpha = -p(x + y\cos 2\theta)i_\alpha$$

$$p\left(\frac{x_0}{2} + x + y\cos 2\theta\right)i_\alpha = -\sin\theta$$

Integrating both sides of the above equation, and solving for $i_\alpha$,

$$i_\alpha = \frac{2(\cos\theta - \cos\theta_0)}{x_0 + 2x + 2y\cos 2\theta} \tag{47}$$

Differentiating, we have

$$p i_\alpha = \frac{-2\sin\theta(x_0 + 2x + 2y\cos 2\theta) + 8y\sin 2\theta(\cos\theta - \cos\theta_0)}{(x_0 + 2x + 2y\cos 2\theta)^2}$$

$$\begin{aligned}e_\alpha &= \tfrac{1}{2}x_0 p i_\alpha\\ &= -x_0\frac{(x_0 + 2x + 2y\cos 2\theta)\sin\theta - 4y\sin 2\theta(\cos\theta - \cos\theta_0)}{(x_0 + 2x + 2y\cos 2\theta)^2}\end{aligned} \tag{48}$$

$$\begin{aligned}\Delta e_\beta &= -py\sin 2\theta i_\alpha = -y\sin 2\theta p i_\alpha - 2y i_\alpha\cos 2\theta\\ &= 2y\frac{\left\{\begin{matrix}(x_0 + 2x + 2y\cos 2\theta)\sin\theta\sin 2\theta - \\ [4y + 2(x_0 + 2x)\cos 2\theta](\cos\theta - \cos\theta_0)\end{matrix}\right\}}{(x_0 + 2x + 2y\cos 2\theta)^2}\end{aligned} \tag{49}$$

Before the fault, from [30] with $I_f = 1$, $e_\beta = \cos\theta$. After the fault,

$$e_\beta = \cos\theta + \Delta e_\beta \qquad [50]$$

where $\Delta e_\beta$ is given by [49].

From [14] and [15] of Chapter I, with instantaneous voltages replacing rms values, and $e_0 = -e_\alpha$ from [46], $e_b$ and $e_c$ are

$$e_b = -\frac{3}{2}e_\alpha + \frac{\sqrt{3}}{2}e_\beta = -\frac{3}{2}e_\alpha + \frac{\sqrt{3}}{2}(\cos\theta + \Delta e_\beta) \qquad [51]$$

$$e_c = -\frac{3}{2}e_\alpha - \frac{\sqrt{3}}{2}e_\beta = -\frac{3}{2}e_\alpha - \frac{\sqrt{3}}{2}(\cos\theta + \Delta e_\beta) \qquad [52]$$

where $e_\alpha$ is given by [48] and $\Delta e_\beta$ by [49].

The maximum values of $e_b$ and $e_c$ given by [51] and [52], respectively, depend upon both $e_\alpha$ and $e_\beta$. $e_\alpha$ depends largely upon $x_0$. From [48], if $x_0 = 0$, $e_\alpha = 0$; if $x_0 = \infty$, $e_\alpha = -\sin\theta$.

*Let* $x_0 = \infty$. Then $i_\alpha = 0$ from [47], and $\Delta e_\beta = 0$ from [49]. From [48], $e_\alpha = -\sin\theta$; and from [46], $e_0 = -e_\alpha = \sin\theta$. From [51] and [52],

$$e_b = \frac{3}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \sqrt{3}\sin(\theta + 30°)$$

$$e_c = \frac{3}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = \sqrt{3}\sin(\theta - 30°)$$

This is the familiar case of a ground fault on an isolated neutral system in which the voltages to ground of the unfaulted phases are $\sqrt{3}$ times normal line-to-neutral voltage.

*Let* $x_0 = 0$. Then from [48], $e_\alpha = 0$. For this case, $e_b$ and $e_c$ will have their maximum values when $e_\beta$ is maximum. The case of maximum flux linkages in the $\alpha$ circuit at the instant of fault will be considered.

*Maximum flux linkages* in the $\alpha$ circuit at time $t = 0$ occurs when $\theta_0 = 0$ or $\pi$; see [29]. With $\theta_0 = 0$, the value of $\theta$ at which $e_\beta$ in [50] is maximum or minimum can be determined by substituting $\Delta e_\beta$ from [49] in [50] and equating the derivative of $e_\beta$ with respect to $\theta$ to zero. If this is done, it will be seen that $\sin\theta = 0$. Therefore $\theta = 0$ or $\pi$. The maximum value occurs at $\theta = \pi$ and the minimum value at $\theta = 0$.

With $\theta = \theta_0 + t = 0 + t = \pi$, maximum value occurs one-half cycle after the instant of fault. At $\theta = \pi$ and $\theta_0 = 0$,

$$e_\beta = -1 + \frac{4y(4y + 4x)}{(2x + 2y)^2} = -1 + \frac{4y}{x + y}$$

If $x$ and $y$ are replaced by their values from [40], the above equation becomes

$$e_\beta = -1 + \frac{2(x_d'' - x_q'')}{x_d''} = 1 - 2\frac{x_q''}{x_d''} \qquad [53]$$

If [53] is substituted in [51] and [52] with $e_\alpha = 0$,

$$e_b = \sqrt{\frac{3}{2}}\left(1 - 2\frac{x_q''}{x_d''}\right) \qquad [54]$$

$$e_c = \frac{\sqrt{3}}{2}\left(2\frac{x_q''}{x_d''} - 1\right) \qquad [55]$$

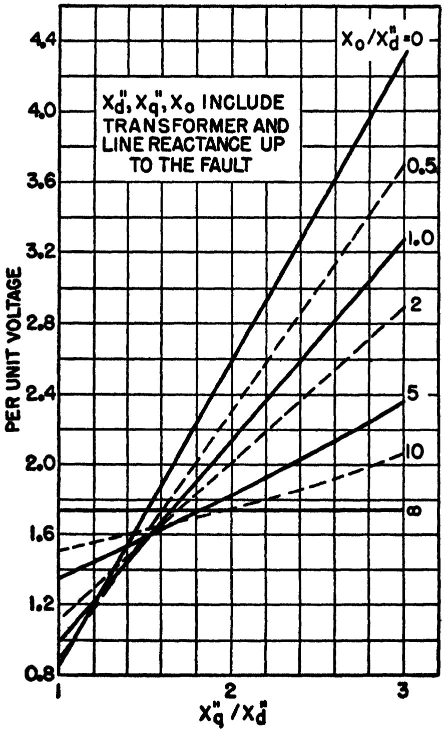

FIG. 3. Line-to-ground fault — maximum voltage of the unfaulted phases in per unit of crest voltage before the fault with resistance and capacitance neglected.

If $x_q'' = x_d''$ the voltage of the unfaulted phases is reduced to $\sqrt{3}/2$ times normal line-to-neutral voltage. If $x_q'' = 2x_d''$, $|e_b| = |e_c| = (\sqrt{3}/2) \times 3 = 2.60$ times normal line-to-neutral peak voltage at maximum flux linkage, neglecting decrements.

Maximum voltages of the unfaulted phases for a line-to-ground fault versus $x_q''/x_d''$ with $x_0/x_d''$ as parameter are given in Fig. 3. With $x_0/x_d'' = 0$ or $\infty$, the equations are given above. (It is suggested that the student determine the value of $\theta$ at which $e_b$ or $e_c$ is maximum for values of $x_0$ between 0 and $\infty$, and check the curves of Fig. 3.)

**Double Line-to-Ground Short Circuit on Phases *b* and *c*.** This case is left for solution by the student (solved in reference 10).

**Harmonic and Natural-Frequency Components.** Equations [42] and [50] in the general case include d-c components, even- and odd-harmonic components and fundamental-frequency terms. The fundamental-frequency components can be determined with a good degree of precision by the method of symmetrical components. Even harmonics (including d-c components) result from flux trapped in the faulted phase or phases and are attenuated by the time constant of the armature. (See [31], Chapter VII.) In unbalanced faults on systems with appreciable capacitance, there will also be natural-frequency components which correspond to the natural frequencies of the loops formed by the unfaulted phase or phases. Natural-frequency components, like even-harmonic components, are attenuated by armature time constants. Odd harmonics are attenuated by rotor time constants. Odd harmonics are present after even harmonics and natural-frequency terms have disappeared. They are present as long as the unbalanced short circuit remains on the system because negative-sequence currents flowing in the armature of a synchronous machine in which $x_d''$ and $x_q''$ are unequal generate positive-sequence third harmonic voltages which, with the unbalanced fault on the system, produce third harmonic positive- and negative-sequence currents. Third harmonic negative-sequence currents generate fifth harmonic positive-sequence voltages, which produce fifth harmonic positive- and negative-sequence currents. It will be shown that odd-order harmonic negative-sequence currents generate positive-sequence voltages of the next higher odd-harmonic order, whereas positive-sequence odd-harmonic currents above the fundamental generate negative-sequence voltages of the next lower odd-harmonic order.

The highest overvoltages are obtained with unbalanced faults on open lines with capacitance. These voltages may be extremely high if there are third harmonic resonance and no limiting effects such as corona, loads, or voltage arresters.

From the voltage curves obtained on the differential analyzer in reference 7, for unbalanced faults on open lines with capacitance it can be seen that the voltages in the first few cycles, if the decay of field flux linkages is neglected, are no higher than the values during later cycles after even harmonics and natural-frequency terms have become negligible. This is because some of the components initially oppose each other. In the region where the highest overvoltages occur (near third harmonic resonance) voltages require several cycles to build up to their peak values which occur after armature transients (even-harmonic and natural-frequency terms) have become negligible but odd harmonics remain. In such cases, or in any case where there are synchronous machines in which $x_d'' \neq x_q''$, it is of advantage to have a simple method for determining overvoltages resulting from fundamental-frequency and odd-harmonic terms.

## THE METHOD OF SYMMETRICAL COMPONENTS APPLIED TO THE DETERMINATION OF ODD HARMONICS

This method is especially useful in estimating the magnitude of third harmonic voltages which, under certain conditions, have been high enough to damage equipment during unbalanced short circuits. Before the method is described, a notation will be adopted and equations derived for determining the voltages produced by positive- and negative-sequence currents of odd-harmonic order.

**Notation for Odd-Harmonic Sinusoidal Currents.** In calculations involving sinusoidal currents and voltages of fundamental frequency, it is customary to use the voltage of phase *a* (or any other specified voltage or current) as reference vector. An analogous procedure will be followed in calculations involving instantaneous currents where fundamental frequency and odd harmonics only are to be considered. Phase *a* will be reference phase. The voltage $e_a$ generated in phase *a* by d-c field current $I_f$ will be reference for all odd-harmonic voltages and currents. Let $e_a$ be written

$$e_a = -I_f \sin\theta \; [=]^* \; I_f \underline{/\theta} \qquad [56]$$

where $I_f \underline{/\theta}$ is a vector revolving at synchronous speed, $I_f$ is the d-c field current, assumed constant, and $\theta$ is the position of the axis of the rotor measured in the forward direction from the axis of phase *a*. [=] indicates that the sinusoidal quantity on one side of it is represented by the revolving vector on the other side.

Let

$e_{na1}$ = positive-sequence *n*th harmonic component of voltage of phase *a*

$e_{na2}$ = negative-sequence *n*th harmonic component of voltage of phase *a*

$i_{na1}$ = positive-sequence *n*th harmonic component of current of phase *a*

$i_{na2}$ = negative-sequence *n*th harmonic component of current of phase *a*

where the first subscript *n* indicates the harmonic order (1, 3, 5, 7, etc.), the second subscript the reference phase *a*, and the third the sequence — 1 for positive and 2 for negative.

With $e_a$ given by [56] as reference, positive- and negative-sequence fundamental-frequency and *n*th harmonic components of voltage of

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[*] The symbol [=], as used here, is not established notation.

phase $a$ will be written

$$\begin{aligned} e_{1a1} &= -E_{1a1} \sin (\theta + \phi_{1a1}) \; [=] \; E_{1a1}\underline{/\theta + \phi_{1a1}} \\ e_{1a2} &= -E_{1a2} \sin (\theta + \phi_{1a2}) \; [=] \; E_{1a2}\underline{/\theta + \phi_{1a2}} \\ e_{na1} &= -E_{na1} \sin (n\theta + \phi_{na1}) \; [=] \; E_{na1}\underline{/n\theta + \phi_{na1}} \\ e_{na2} &= -E_{na2} \sin (n\theta + \phi_{na1}) \; [=] \; E_{na2}\underline{/n\theta + \phi_{na2}} \end{aligned} \qquad [57]$$

where $E_{na1}$ indicates the crest of the positive-sequence sinusoidal voltage $e_{na1}$ and $E_{na2}$ that of the negative-sequence sinusoidal voltage $e_{na2}$; $\phi_{na1}$ and $\phi_{na2}$ (which may have any value between 0° and 360°) give the phase relative to $e_a$ at the instant $\theta = 0$.

Let

$$\begin{aligned} i_{na1} &= -I_{na1} \sin \left( n\theta + \gamma_{na1} - \frac{\pi}{2} \right) = I_{na1} \cos (n\theta + \gamma_{na1}) \\ i_{na2} &= -I_{na2} \sin \left( n\theta + \delta_{na2} - \frac{\pi}{2} \right) = I_{na2} \cos (n\theta + \delta_{na2}) \end{aligned} \qquad [58]$$

where $n$ indicates the odd-harmonic order (1, 3, 5, etc.), and $\gamma_{na1}$ and $\delta_{na2}$ may have any value between 0° and 360°. For convenience, positive- and negative-sequence currents are taken $\pi/2$ radians or 90° behind their respective voltages when $\gamma$ and $\delta$ are zero.

The operator $\pm j$ applied to a vector turns it through ±90°. If the revolving vector representing an instantaneous sinusoidal current or voltage is multiplied by $\pm j$, its angular position is increased by ±90° relative to the revolving reference vector $I_f\underline{/\theta}$. The new revolving vector now represents a new instantaneous sinusoidal quantity which is ±90° ahead of the initial one in phase.

Let an instantaneous sinusoidal voltage $e$ be written

$$e = -E_m \sin (\theta + \alpha) \; [=] \; E_m\underline{/\theta + \alpha}$$

If the vector $E_m\underline{/\theta + \alpha}$ is multiplied by $\pm j$, there results

$$\pm jE_m\underline{/\theta + \alpha} = E_m\underline{/\theta + \alpha \pm 90^\circ} \; [=] \; - E_m \sin (\theta + \alpha \pm 90^\circ)$$

The new instantaneous voltage $-E_m \sin (\theta + \alpha \pm 90^\circ)$ is ±90° ahead of the initial instantaneous voltage $e$.

**Symbols [j]* and [−j].*** Let the symbol [$j$] indicate that the phase of the instantaneous sinusoidal quantity which it precedes is to be advanced by 90°; let [$-j$] indicate that it is to be retarded by 90°.

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* This is not established notation.

Thus

$$\begin{aligned} [j]e &= [j][-E_m \sin(\theta+\alpha)] = -E_m \sin\left(\theta+\alpha+\frac{\pi}{2}\right) \\ &= -E_m \cos(\theta+\alpha) \\ [-j]i_{na1} &= [-j]I_{na1} \cos(n\theta+\gamma) = I_{na1} \cos\left(n\theta+\gamma-\frac{\pi}{2}\right) \\ &= I_{na1} \sin(n\theta+\gamma) \end{aligned} \tag{59}$$

NOTE : $[j]$ and $[-j]$ are to be applied only to sinusoidal quantities.

### Voltages Produced by Positive- and Negative-Sequence Currents of Odd-Harmonic Order in Machines in which $x_d'' \neq x_q''$

Equations [38] and [39] will be used as the connecting link between symmetrical components and direct- and quadrature-axis components. From equations [25]–[32] of Chapter I, with instantaneous voltages and currents replacing vector values,

$$\begin{aligned} i_\alpha &= i_{a1} + i_{a2} \\ i_\beta &= -[j](i_{a1} - i_{a2}) \end{aligned} \qquad [60]$$

$$\begin{aligned} e_{a1} &= \tfrac{1}{2}(e_\alpha + [j]e_\beta) \\ e_{a2} &= \tfrac{1}{2}(e_\alpha - [j]e_\beta) \end{aligned} \qquad [61]$$

The above equations containing $[j]$ are applicable only to sinusoidal quantities.

Equations [60] express $i_\alpha$ and $i_\beta$ in terms of fundamental-frequency positive- and negative-sequence currents. The changes in $\alpha$ and $\beta$ currents due to positive- and negative-sequence currents of the $n$th harmonic order are

$$\begin{aligned} \Delta i_\alpha &= i_{na1} + i_{na2} \\ \Delta i_\beta &= -[j](i_{na1} - i_{na2}) \end{aligned} \qquad [60a]$$

The changes in positive- and negative-sequence voltages in terms of $\Delta e_\alpha$ and $\Delta e_\beta$ are

$$\begin{aligned} \Delta e_{a1} &= \tfrac{1}{2}(\Delta e_\alpha + [j]\Delta e_\beta) \\ \Delta e_{a2} &= \tfrac{1}{2}(\Delta e_\alpha - [j]\Delta e_\beta) \end{aligned} \qquad [61a]$$

If $\Delta e_\alpha$ and $\Delta e_\beta$ in [61a] are of fundamental frequency, $\Delta e_{a1}$ and $\Delta e_{a2}$ will also be of fundamental frequency; but if $\frac{1}{2}(\Delta e_\alpha + [j]\Delta e_\beta)$ and $\frac{1}{2}(\Delta e_\alpha - [j]\Delta e_\beta)$ contain harmonics, $\Delta e_{a1}$ and $\Delta e_{a2}$ may also have harmonics. Let $\Delta e_{a1}$ and $\Delta e_{a2}$ therefore represent positive- and negative-sequence voltages, respectively, of indeterminate harmonic order until this order has been determined.

*Positive-sequence currents of fundamental frequency* produce fundamental mmf space waves in the direct and quadrature axes which are stationary with respect to the rotor. Direct-axis positive-sequence currents meet direct-axis reactance, and quadrature-axis positive-sequence currents meet quadrature-axis reactance. This is explained in Chapter VII. The following development is merely a test of [38] and [39].

Assume first that the positive-sequence currents are suddenly applied direct-axis currents which produce flux linkage in the armature windings opposite to those produced by the main field winding. This current, $i_{1a1}$, which lags by $\pi/2$ radians the generated voltage $e_a$ given by [56], may be written

$$i_{1a1} \quad -I_{1a1} \sin\left(\theta - \frac{\pi}{2}\right) = I_{1a1} \cos\theta \; [=] \; I_{1a1} \angle \underline{\theta - \frac{\pi}{2}}$$

The mmf and flux linkages in the armature due to this current oppose those due to the field current $I_f$; this is taken into account by the negative signs in [32] and [33] and in [38] and [39] derived from them.

$$\Delta i_\alpha = i_{1a1} = I_{1a1} \cos\theta = I_1 \cos\theta$$

$$\Delta i_\beta = -[j] i_{1a1} = I_{1a1} \sin\theta = I_1 \sin\theta$$

where $I_1 = I_{1a1}$ represents the crest of the direct-axis sinusoidal current $i_{1a1}$.

When $\Delta i_\alpha$ and $\Delta i_\beta$ are substituted in [38] and [39],

$$\begin{aligned}
\Delta e_\alpha &= -I_1[r + p(x + y\cos 2\theta)]\cos\theta - I_1(py \sin 2\theta \sin\theta) \\
&= -I_1(r + px)\cos\theta - I_1 yp\,(\cos 2\theta \cos\theta + \sin 2\theta \sin\theta) \\
&= -I_1(r + px)\cos\theta - I_1 yp \cos\theta = -I[r + p(x + y)]\cos\theta \\
\Delta e_\beta &= -I_1(py \sin 2\theta \cos\theta) - I_1[r + p(x - y\cos 2\theta)\sin\theta] \\
&= -I_1(r + px)\sin\theta - I_1 yp \sin\theta = -I_1[r + p(x + y)]\sin\theta \\
[j]\Delta e_\beta &= -I_1[r + p(x + y)]\cos\theta
\end{aligned}$$

If $x$ and $y$ in the above equations are replaced by their values from [40], $(x + y) = x_d''$. When $\Delta e_\alpha$ and $[j]\Delta e_\beta$ replace $e_\alpha$ and $[j]e_\beta$ in [61], and $p$ is replaced by $j$,

$$\begin{aligned}
\Delta e_{a1} &= -(r + jx_d'')I_{1a1}\cos\theta = -(r + jx_d'')i_{1a1} \\
\Delta e_{a2} &= 0
\end{aligned} \qquad [62]$$

Equations [62] give expected results: Positive-sequence currents of fundamental frequency produce no negative-sequence voltages; the reactance $x_d''$ is met by suddenly applied positive-sequence currents which lag by 90° the reference voltage $e_a$ given by [56]. In a similar manner, it can be shown that suddenly applied quadrature-axis positive-sequence currents meet quadrature-axis reactance $x_q''$.

*Negative-sequence currents of fundamental frequency*, whether suddenly applied or sustained, produce a fundamental mmf space wave which rotates backward at twice synchronous speed relative to the rotor. The conditions which govern the voltages produced by them are therefore always subtransient, regardless of how long an unsymmetrical fault remains on the system. Therefore, [38]–[40] can be used to determine these voltages.

In accordance with the notation adopted in [58],

$$i_{1a2} = -I_{1a2} \sin\left(\theta + \delta_{1a2} - \frac{\pi}{2}\right) = I_2 \cos(\theta + \delta)$$

where (for simplification in the development) $I_2$ and $\delta$ replace $I_{1a2}$ and $\delta_{1a2}$, respectively.

$$\begin{aligned} \Delta i_\alpha &= i_{1a2} = I_2 \cos(\theta + \delta) \\ \Delta i_\beta &= [j]i_{1a2} = -I_2 \sin(\theta + \delta) \end{aligned}$$

When $\Delta i_\alpha$ and $\Delta i_\beta$ are substituted in [38] and [39],

$$\begin{aligned} \Delta e_\alpha &= -I_2(r + px) \cos(\theta + \delta) \\ &\quad - I_2 py [\cos 2\theta \cos(\theta + \delta) - \sin 2\theta \sin(\theta + \delta)] \\ &= -I_2(r + px) \cos(\theta + \delta) - I_2 py \cos(3\theta + \delta) \\ &= -I_2(r + jx) \cos(\theta + \delta) + 3I_2 y \sin(3\theta + \delta) \\ \Delta e_\beta &= I_2(r + px) \sin(\theta + \delta) \\ &\quad - I_2 py [\sin 2\theta \cos(\theta + \delta) + \cos 2\theta \sin(\theta + \delta)] \\ &= I_2(r + jx) \sin(\theta + \delta) - 3I_2 y \cos(3\theta + \delta) \\ [j]\Delta e_\beta &= I_2(r + jx) \cos(\theta + \delta) + 3I_2 y \sin(3\theta + \delta) \end{aligned}$$

Substitution of $\Delta e_\alpha$ and $[j]\Delta e_\beta$ in [61*a*] gives

$$\begin{aligned} \Delta e_{a1} &= 3I_2 y \sin(3\theta + \delta) = -3I_2(-y) \sin(3\theta + \delta) \\ &= 3I_2(-y) \cos\left(3\theta + \delta + \frac{\pi}{2}\right) = e_{3a1} \end{aligned}$$

$$e_{3a1} = 3I_{1a2} \frac{x_q'' - x_d''}{2} \cos\left(3\theta + \delta_{1a2} + \frac{\pi}{2}\right) \qquad [63]$$

$$\Delta e_{1a2} = -(r + jx)I_2 \cos(\theta + \delta) = -\left(r + j\frac{x_d'' + x_q''}{2}\right)i_{1a2} \qquad [64]$$

**Negative-Sequence Reactance $x_2$.** For an ideal synchronous machine, which is here assumed, the negative-sequence reactance $x_2$, defined* as the ratio of the fundamental-frequency negative-sequence reactive armature voltage produced by negative-sequence current of fundamental frequency to this current, from [64], is

$$x_2 = \frac{x_d'' + x_q''}{2} \qquad [65]$$

In an actual machine, $x_2$ is given approximately by [65]. In a salient-pole machine, without an amortisseur or its equivalent, $x_d'' = x_d'$ and $x_q'' = x_q' = x_q$.

**Positive-Sequence Third Harmonic Voltages Produced by Negative-Sequence Currents.** From [63], negative-sequence current of fundamental frequency produces a positive-sequence third harmonic voltage $e_{3a1}$ of magnitude $3I_2(-y) = 3I_2[(x_q'' - x_d'')/2]$, which (when $x_q'' > x_d''$) is 90° or $\pi/2$ radians ahead of $i_{1a2}$ at the instant $\theta = 0$. Its phase at $\theta = 0$ relative to the reference voltage $e_a$ given by [56] can be determined when $\delta_2$ is known. Thus, if $i_{1a2}$ lags $e_a$ by 90°, $\delta_2 = 0$ and $e_{3a1}$ is in phase at $\theta = 0$ with $e_a$; but if $i_{1a2}$ leads $e_a$ by 90°, $\delta_2 = 180°$ and $e_{3a1}$ is 180° out of phase with $e_a$ at $\theta = 0$.

**Negative-Sequence Currents of Odd-Harmonic Order above the First.** The fundamental space wave of mmf due to third harmonic negative-sequence currents rotates backward at three times synchronous speed relative to the armature; it therefore rotates backward at four times synchronous speed relative to the rotor. Similarly the fundamental mmf space wave due to negative-sequence currents of the $n$th odd-harmonic order rotates backward at $n + 1$ times synchronous speed relative to the rotor. The conditions in the machine are therefore subtransient, and [38]–[40] apply. By a development similar to that for negative-sequence currents of fundamental frequency, it can be shown that negative-sequence currents $i_{na2}$ of any odd-harmonic order $n$ generate positive-sequence voltages of the next higher odd-harmonic order $(n + 2)$ of magnitude $n(-y)I_{na2}$ which (when $x_q'' > x_d''$) lead in phase by 90° at $\theta = 0$ the currents which produce them.

It can likewise be shown that the reactance met by the negative-sequence $n$th harmonic current is $nx_2$ where $x_2$ is given by [65].

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* For other definitions of $x_2$ see " The Negative-Sequence Reactances of an Ideal Synchronous Machine," by William C. Duesterhoeft, *A.I.E.E. Trans.*, Vol. 68, 1949.

With negative-sequence currents $i_{na2}$ of the $n$th odd-harmonic order in the machine,

$$\begin{aligned} i_{na2} &= I_{na2} \cos (n\theta + \delta_{na2}) \\ e_{(n+2)a1} &= (n+2)(-y) I_{na2} \cos [(n+2)\theta + \delta_{na2} + 90°] \\ &= (n+2)\left(\frac{x_q'' - x_d''}{2}\right) I_{na2} \cos [(n+2)\theta + \delta_{na2} + 90°] \end{aligned} \qquad [66]$$

$$x_{n2} = nx = n\frac{x_d'' + x_q''}{2} \qquad [67]$$

It will be noted that the magnitude of the positive-sequence harmonic voltage given by [66] varies directly with the magnitude $I_{na2}$ of the negative-sequence current producing it. This current is subject to attenuation by rotor time constants but as long as an unbalanced short circuit remains on the system (as long as there are negative-sequence currents) positive-sequence odd-harmonic voltages will be generated.

**Negative-Sequence Voltages Generated by Positive-Sequence Currents of Third or Higher Harmonic Order.** Let the positive-sequence third harmonic current $i_{3a1} = I_3 \cos (3\theta + \gamma)$. Then

$$\begin{aligned} \Delta i_\alpha &= i_{3a1} = I_3 \cos (3\theta + \gamma) \\ \Delta i_\beta &= -[j] i_{3a1} = I_3 \sin (3\theta + \gamma) \end{aligned}$$

From [38] and [39], where $p = d/dt$,

$$\begin{aligned} \Delta e_\alpha &= -I_3(r + px) \cos (3\theta + \gamma) - I_3 py [\cos 2\theta \cos (3\theta + \gamma) \\ &\qquad + \sin 2\theta \sin (3\theta + \gamma)] \\ &= -I_3(r + px) \cos (3\theta + \gamma) - I_3 py \cos (\theta + \gamma) \\ &= -I_3(r + j3x) \cos (3\theta + \gamma) - I_3 y \cos \left(\theta + \gamma + \frac{\pi}{2}\right) \\ \Delta e_\beta &= -I_3(r + px) \sin (3\theta + \gamma) - I_3 py [\sin 2\theta \cos (3\theta + \gamma) \\ &\qquad - \cos 2\theta \sin (3\theta + \gamma)] \\ &= -I_3(r + px) \sin (3\theta + \gamma) + I_3 py \sin (\theta + \gamma) \\ &= -I_3(r + j3x) \sin (3\theta + \gamma) + I_3 y \cos (\theta + \gamma) \end{aligned}$$

$$[j]\Delta e_\beta = -I_3 \left[(r + j3x) \cos (3\theta + \gamma) - y \cos \left(\theta + \gamma + \frac{\pi}{2}\right)\right]$$

Substitution of $\Delta e_\alpha$ and $[j]\Delta e_\beta$ in [61*a*] gives

$$\Delta e_{a1} = -I_3(r + j3x)\cos(3\theta + \gamma) = -(r + j3x)i_{3a1}$$

$$= \left[-r - j3\left(\frac{x_d'' + x_q''}{2}\right)\right] i_{3a1} = \Delta e_{3a1} \qquad [68]$$

$$\Delta e_{a2} = -I_3 y \cos\left(\theta + \gamma + \frac{\pi}{2}\right) = I_3(-y)\cos\left(\theta + \gamma + \frac{\pi}{2}\right)$$

$$= I_3 \frac{x_q'' - x_d''}{2} \cos\left(\theta + \gamma + \frac{\pi}{2}\right) = e_{1a2} \qquad [69]$$

From [68], third harmonic positive-sequence currents flowing in the machine meet the reactance $3x = 3[(x_d'' + x_q'')/2]$. From [69], third harmonic positive-sequence currents generate negative-sequence voltages of magnitude $I_{3a1}(-y) = I_{3a1}[(x_q'' - x_d'')/2]$ which (when $x_q'' > x_d''$) lead by 90° at $\theta = 0$ the currents which produce them. By a similar development, it can be shown that positive-sequence currents $I_{na1}$ of any higher odd-harmonic order $n$ generate negative-sequence voltages of the next lower odd-harmonic order $(n - 2)$ of magnitude $[(-y)(n - 2)I_{na1}]$ which (when $x_q'' > x_d''$) lead by 90° at $\theta = 0$ the currents which produce them. The reactance met by positive-sequence current of odd-harmonic order $n$ above the first is $nx_2$. If

$$i_{na1} = I_{na1}\cos(n\theta + \gamma_{na1}), \qquad [70]$$

$$e_{(n-2)a2} = [-y(n - 2)I_{na1}]\cos\left[(n - 2)\theta + \gamma_{na1} + \frac{\pi}{2}\right] \qquad [71]$$

$$x_{n1} = nx_2 = n\left(\frac{x_d'' + x_q''}{2}\right)$$

where $n$ is odd and greater than unity.

**Method of Symmetrical Components.** When the method of symmetrical components is applied to the determination of fundamental-frequency voltages and currents during balanced and unbalanced short circuits, direct-axis reactances are customarily used for the positive-sequence reactances of synchronous machines. Thus $x_d''$, $x_d'$, and $x_d$ are used for the positive-sequence reactance $x_1$ of the machine under initial, transient, and sustained conditions, respectively. This is strictly correct only when the positive-sequence currents in the machine are entirely direct-axis currents or the reactances in the two axes are equal. In the usual case, positive-sequence currents resulting from short circuits are largely direct-axis currents, so that the error in using only direct-axis reactance in such cases is relatively unimportant. Moreover, the rapidity and ease of solution of short-circuit problems would be greatly impaired if two reactances were used for the positive-sequence reactances of all synchronous machines in a system instead of only their direct-axis reactances. The method as conventionally applied neglects the effect upon fundamental-frequency currents and voltages of the voltages generated by positive- and negative-sequence currents in machines in which $x_d'' \neq x_q''$; for example, the negative-sequence voltages of fundamental frequency generated by positive-sequence third harmonic currents. However, such effects are small, and the error in neglecting them does not appreciably affect calculated fundamental-frequency voltages and currents in the usual short-circuit problem. A possible exception may occur at or near third harmonic resonance.

## Determination of Odd-Harmonic Voltages Resulting from Unbalanced Faults

(1) In a system of any complexity, calculate fundamental-frequency voltages at any desired system points by the conventional method of symmetrical components, using the generated voltage of phase as a reference vector. Include in these calculations the negative-sequence currents in all synchronous machines in which $x_d'' \neq x_q''$.

(2) From [66], calculate the third harmonic positive-sequence voltages in magnitude and phase generated in all synchronous machines in which $x_d'' \neq x_q''$ by the negative-sequence currents of fundamental frequency calculated under 1.

(3) With the generated third harmonic voltages calculated under 2 acting in the synchronous machines of the system in which $x_d'' \neq x_q''$, determine by the method of symmetrical components the third harmonic voltages at any desired system points under given fault conditions, using three times fundamental inductive reactances and one-third fundamental capacitive reactances. As an approximation, use $3x_2$ (three times the negative-sequence reactance) for the reactance of synchronous machines to both positive- and negative-sequence third harmonic currents. Include in these calculations the third harmonic currents in all synchronous machines in which $x_d'' \neq x_q''$.

(4) From [66], calculate the fifth harmonic voltages in magnitude and phase generated in all synchronous machines in which $x_d'' \neq x_q''$ by the third harmonic negative-sequence currents calculated under 3.

(5) With the generated fifth harmonic voltages calculated under 4, proceed as in 3, using fifth harmonic reactances instead of third.

By proceeding in this manner until increments of voltage become negligible, positive-sequence odd-harmonic voltages generated by " first-calculated " negative-sequence currents can be determined in magnitude and phase relative to the reference vector and superposed at any value of $\theta$ to obtain resultant positive-sequence voltage at any desired system point. " First-calculated " negative-sequence currents refer to those calculated under 1, 3, 5, etc., above.

It will be noted from [69] and [71] that positive-sequence third harmonic currents generate negative-sequence voltages of fundamental frequency, and positive-sequence fifth harmonic currents generate negative-sequence voltages of third harmonic frequency, each positive-sequence harmonic current of odd order higher than the first generating a negative-sequence voltage of the next lower odd-harmonic order. The combined effects, and resultant effects, on system overvoltages of these generated voltages will in general be small relative to those produced by " first-calculated " negative-sequence currents. Neglecting them will not greatly alter the magnitude of the calculated voltages in the usual system where resistances, capacitances, and inductances are present. And to include them greatly impairs the rapidity and ease of solution by this method. To illustrate:

The fifth harmonic negative-sequence voltages generated by positive-sequence seventh harmonic currents, with an unbalanced short circuit on the system, produce positive- and negative-sequence fifth harmonic currents. The former generate negative-sequence third harmonic voltages, the latter positive-sequence seventh harmonic voltages. These voltages, with the fault on the system, produce positive- and negative-sequence currents of their respective harmonic order which in turn generate negative-sequence voltages of lower harmonic order and positive-sequence voltages of higher harmonic order. To include the total effect upon the resultant calculated voltage at any specified system point of all positive- and negative-sequence components of voltage resulting from " first-calculated " positive-sequence harmonic currents would be laborious and, in general, unnecessary. A possible exception might occur with third harmonic resonance.* In this case, fundamental-frequency voltages calculated by the conventional method of symmetrical components would also be in error.

*Even harmonics and natural-frequency components* of current and voltage are not included in this symmetrical component method. When these components, which are attenuated by armature time

* The need for further study of this subject is indicated.

constants, are present, they do not affect the magnitudes of the initial fundamental-frequency and odd-harmonic components of voltage, but they do affect the resultant voltages which contain all components. It is only after armature transients, when present initially, have become negligible that resulting system voltages consist only of odd-order harmonics; and at this time, the currents induced in amortisseur windings by suddenly applied positive-sequence currents will have become negligible. If voltages after the first few cycles following an unbalanced short circuit are to be determined, direct-axis transient reactance $x_d'$ should be used for positive-sequence reactance $x_1$ in determining fundamental-frequency currents; under sustained conditions, $x_d$ is used for $x_1$. Negative-sequence reactance $x_2$, however, is $x_2 = \frac{1}{2}(x_d'' + x_q'')$; and the reactance to $n$th harmonic positive- and negative sequence currents is $nx_2$, regardless of how long the short circuit remains on the system. Also, the reactance $-y = \frac{1}{2}(x_q'' - x_d'')$ always applies in determining harmonic voltages generated in a machine in which $x_d'' \neq x_q''$ by negative-sequence currents. See [66]. In a salient-pole machine without an amortisseur winding, $x_d'' = x_d'$ and $x_q'' = x_q' = x_q$. In the special case of no even harmonics and no natural-frequency components, such as will occur at zero flux linkages in the faulted loop at $t = 0$ and negligible capacitance in the system, only odd-harmonic components of voltage will be present. In such cases maximum voltages during the first half cycle can be approximately determined by using $x_d''$ for positive-sequence reactance $x_1$ in a machine with an amortisseur in determining fundamental-frequency currents and voltages. In a salient-pole machine without an amortisseur winding, $x_d'$ is used for $x_1$ during the initial cycle as well as in later cycles.

**Symmetrical Three-Phase System under Previously Balanced Operation.** Consider, for example, that power is being supplied from a synchronous generator in which $x_d'' \neq x_q''$ through transformers to a system of appreciable capacitance in which positive- and negative-sequence impedances can be assumed equal, and that an unbalanced short circuit occurs at $F$ on the line side of transformers. The problem is to determine the phase voltages to ground at $F$ for various types of short circuits. The first step in the procedure is to determine the fundamental-frequency phase voltages at $F$ and the negative-sequence current $I_{a2}$ in the generator by the conventional method of symmetrical components, using the internal voltage of phase $a$ as reference vector.

**Positive-, Negative-, and Zero-Sequence Harmonic Impedance Diagrams.** When the unbalanced fault occurs at the terminals of the generator-end transformer, it is convenient to reduce harmonic impedance diagrams to two impedances viewed from $F$, the point of fault: one toward the transformer and one toward the external system. Under the assumption that positive- and negative-sequence $n$th harmonic currents meet the same impedance $Z_n = r_2 + jnx_2$ in a synchronous machine and that positive- and negative-sequence system harmonic impedances are equal, positive- and negative-sequence harmonic impedance diagrams will be the same for any odd harmonic $n$, where $n$ is greater than unity.

Let

$r + px_2$ = positive- and negative-sequence impedance of generator and transformer in series, viewed from $F$

$r_0 + px_0$ = zero-sequence impedance viewed from $F$ toward the transformer

$Z_{1e}(p)$ = positive- and negative-sequence impedance viewed from $F$ toward external system

$Z_{0e}(p)$ = zero-sequence impedance viewed from $F$ toward external system

where $x_2$ and $x_0$ are fundamental-frequency reactances, and $p$ preceding them is to be replaced by $jn$ for any odd harmonic $n$ greater than unity. $Z_{1e}(p)$ and $Z_{0e}(p)$ are calculated separately for each odd harmonic. The determination of $Z_{1e}(p)$ and $Z_{0e}(p)$ will be discussed later. For the present let it be assumed that they are known, or can be determined, for any given harmonic.

**Connection of Component Networks to Represent Unbalanced Short Circuits.** Under the assumption of identical positive- and negative-sequence impedance networks for a given harmonic, $\alpha$ and $\beta$ impedance networks are also identical, each being the same as the positive-sequence network. The connection of the component network for determining currents and voltages of a specified harmonic order is analogous to that given in Chapter X of Volume I for determining currents and voltages of fundamental frequency.

For any odd-harmonic order greater than unity, let $i_a$, $i_b$, and $i_c$ be phase currents flowing into the fault. Let $e_a$, $e_b$, and $e_c$ be phase voltages to ground at the fault. Let $e_\alpha$, $e_\beta$, $e_0$ and $i_\alpha$, $i_\beta$, $i_0$ be $\alpha$, $\beta$, and 0 components of voltage and current, respectively, at the fault.

**Line-to-Ground Fault (Phase *a*).** The fault conditions are

$$i_b = i_c = 0; \quad e_a = 0 \qquad [72]$$

From [13]–[18], Chapter I,

$$i_\beta = 0; \quad i_\alpha = 2i_0; \quad e_\alpha = -e_0 \qquad [73]$$

The connection of the $\alpha$ and 0 networks to represent fault conditions is given in Fig. 4, where $r + px_0$ and $Z_{0e}(p)$ are each divided by 2; the $\beta$ network is unaffected by the fault. The applied voltages $e_\alpha(p)$ and $e_\beta(p) = [-j]e_\alpha(p)$ are each equal in magnitude to the positive-sequence voltage of the specified harmonic order generated in the synchronous machine by negative-sequence currents of the next lower odd-harmonic order.

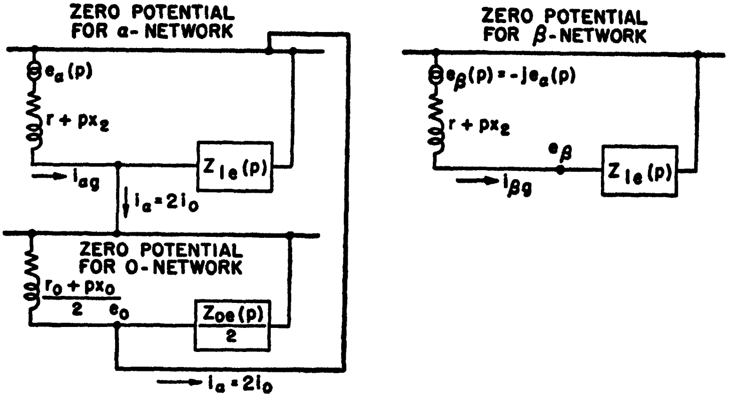

Fig. 4. Equivalent circuit for line-to-ground fault for use in determining harmonic currents and voltages.

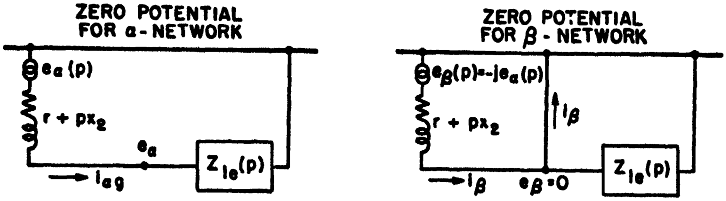

Fig. 5. Equivalent circuit for line-to-line fault for use in determining harmonic currents and voltages.

**Line-to-Line Fault (Phases *b* and *c*).** The fault conditions are

$$i_b = -i_c; \quad i_a = 0; \quad e_b = e_c$$

From [13]–[18], Chapter I,

$$i_\alpha = i_0 = 0; \quad e_\beta = 0 \qquad [74]$$

The $\alpha$ and $\beta$ networks for this condition are shown in Fig. 5; the 0 network is not involved. The $\alpha$ network is unaffected by the fault. The $\beta$ network is short-circuited at the point of fault.

**Double Line-to-Ground Fault (Phase $b$ and $c$).** The fault conditions are

$$i_a = 0; \quad e_b = e_c = 0$$

From [13]–[18], Chapter I,

$$e_\beta = 0; \quad e_\alpha = 2e_0; \quad i_\alpha = -i_0 \qquad [75]$$

The connection of the $\alpha$ and 0 networks for these conditions are shown in Fig. 6, where $r + px_0$ and $Z_{0e}(p)$ are each multiplied by 2. The $\beta$ network is short-circuited at the point of fault.

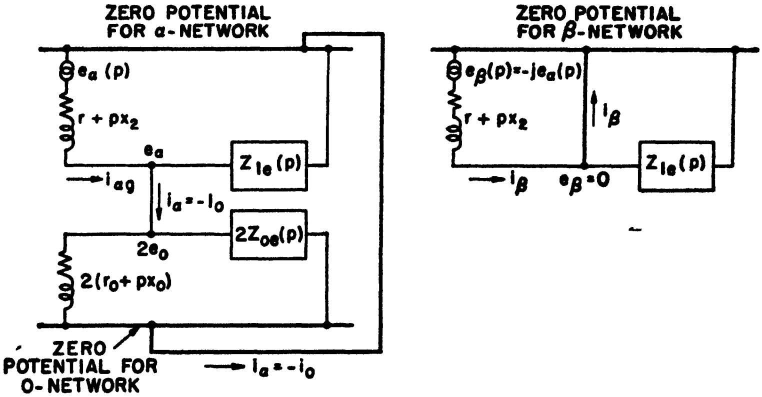

FIG. 6. Equivalent circuit for double-line-to-ground fault for use in determining harmonic currents and voltages.

**Resonance to Harmonics.** From Figs. 4, 5, and 6, resonance to the $n$th harmonic can be determined. For the line-to-ground fault, resonance may occur in the unfaulted $\beta$ circuit or in the $\alpha$ circuit; for the line-to-line or the double-line-to-ground fault, resonance can occur only in the $\alpha$ circuit.

For a line-to-ground fault resonance in the $\alpha$ circuit will occur when

$$nx_2 = \text{negative reactance component of } \frac{r_0 + px_0}{2}, \ \frac{Z_{0e}(p)}{2}$$
$$\text{and } Z_{1e}(p) \text{ in parallel}$$

In the $\beta$ circuit, it will occur when

$$nx_2 = \text{negative reactance component of } Z_{1e}(p)$$

For a line-to-line short circuit, resonance will occur in the unfaulted $\alpha$ circuit when

$$nx_2 = \text{negative reactance component of } Z_{1e}(p)$$

For a double-line-to-ground short circuit resonance will occur in the $\alpha$ circuit when

$$nx_2 = \text{negative reactance component of } 2(r_0 + jx_0),\ 2Z_{0e}(p), \text{ and } Z_{1e}(p) \text{ in parallel}$$

**Harmonic Impedances and Admittances of Transmission Circuits.** Positive- and zero-sequence capacitive susceptances of overhead transmission circuits vary directly with frequency. Positive-sequence inductive reactances of overhead, non-magnetic transmission circuits vary directly with frequency; but this is not true of zero-sequence reactances.

**Zero-Sequence Impedances.** Carson's equations for the self-impedance of a conductor with earth return and the mutual impedance between two conductors with common earth return (Volume I, page 372) apply at all frequencies. If terms in Carson's equations involving height of conductors above ground are neglected, $Z_{00}$, the zero-sequence self-impedance of a three-phase circuit without ground wires, can be determined from [28]–[32], Chapter III, of this volume.

$$\begin{aligned} Z_{00} &= Z_{aa-g} + 2Z_{ab-g} \\ &= r + 3R_g + j(x_i + X_{aa-g} + 2X_{ab-g}) \end{aligned}$$

Expressed in ohms per mile,

$$Z_{00} = r + 0.00477f + jf\left[6.28L_i + 0.0350 + 0.0140 \log_{10} \frac{1}{\sqrt[3]{s^2d}}\sqrt{\frac{\rho}{f}}\right] \qquad [76]$$

where $f$ = frequency in cycles per second
$r$ = resistance of conductor in ohms per mile
$L_i$ = internal inductance of conductor in henries per mile ($x_i = 2\pi fL_i$)
$d$ = diameter of conductor in inches
$s$ = geometric mean spacing between conductors in feet
$\rho$ = earth resistivity in ohms (meter cube).

The terms involving height of conductors above ground neglected in [76] are functions of $f/\rho$. The error in neglecting these terms in calculating 60-cycle zero-sequence reactance is small for values of $\rho$

higher than 10. See Fig. 3(*d*), page 379, Volume I. It will likewise be small in calculating 60-cycle *n*th harmonic zero-sequence reactance if $\rho$ is higher than $10n$. Thus, if $\rho = 100$ or more, [76] can be used with only slight error to calculate 60-cycle harmonic zero-sequence impedances up to the ninth when there are no ground wires. In the case of a given circuit, with or without ground wires, any *n*th harmonic zero-sequence reactance at the given value of $\rho$ will be $n$ times the fundamental-frequency reactance calculated for a value of $\rho$ equal to $1/n$ times the given value of $\rho$.

**Determination of $Z_{1e}(p)$ and $Z_{0e}(p)$.** $Z_{1e}(p)$ and $Z_{0e}(p)$ viewed from the fault, corresponding to any harmonic, can be determined from the equivalent *T*'s (Volume I, Chapter VI) of the transmission circuit with their load ends closed through the corresponding harmonic impedance of the load, including load-end transformer.

*Third harmonics* are the most important of the harmonics resulting from unbalanced short circuits. Third harmonic voltages of the unfaulted phases or phase vary directly with the magnitude of the third harmonic positive-sequence voltage $e_{3a1}$ generated in synchronous machines in which $x_d'' \neq x_q''$ by fundamental-frequency negative-sequence current $I_{a2}$, but they are very much higher than $e_{3a1}$ when there is resonance to third harmonics. For resonance to occur with a line-to-line short circuit ($\alpha$ circuit, Fig. 5) the third harmonic inductive reactance of the load must be higher than the third harmonic positive-sequence capacitive reactance of the line, so that their parallel value and $Z_{1e}(p)$ will be capacitive. This will be the case when the load end of the line is open, and it may occur when the load is very light. Under normal load, this condition is unlikely to exist. For the line-to-ground fault in a solidly grounded system, the conditions for resonance in the unfaulted $\beta$ circuit of Fig. 4 are the same as those for the unfaulted $\alpha$ circuit of Fig. 5. With the line open at the load end, resonance will occur in both circuits if the fundamental-frequency positive-sequence capacitive reactance $x_{c1}$ of the line is approximately 9 times the fundamental-frequency negative-sequence reactance $x_2$ of generator plus transformer viewed from the fault.

With the fault at the terminals of the generating-end transformer, and the system solidly grounded, third harmonic resonance will not be possible in the $\alpha$ circuit of Fig. 4 or 6 for the usual system, because the third harmonic zero-sequence impedance viewed from the fault will normally be inductive. In an ungrounded system, resonance in the $\alpha$ circuit of Fig. 4 for a line-to-ground fault is possible, even for a relatively low third harmonic reactive load impedance. However, in an ungrounded system, the fundamental-frequency negative-sequence current $I_{a2}$ will be small, and the positive-sequence third harmonic voltage $e_{3a1}$ generated by it will likewise be small so that, even with resonance, the resultant third harmonic phase voltages are not likely to be large. For a double-line-to-ground fault in an ungrounded system, $I_{a2}$ is approximately the same as for a line-to-line fault, and resonance is possible in the $\alpha$ circuit of Fig. 6.

The approximate third harmonic resonance regions for various types of faults are given in Fig. 7, with $x_{c1}/x_2$ as abscissa and $x_{1e}/x_2$ as ordinate, where

$x_2$ = fundamental-frequency negative-sequence reactance of generator plus transformer viewed from the fault

$x_{c1}$ = fundamental-frequency positive-sequence capacitive reactance of the transmission line

$x_{1e}$ = fundamental-frequency positive-sequence inductive reactance of load, including load-end transformer

The small $\alpha$ and 0 diagrams to the right of the curves were used to determine the third harmonic values of $Z_{1e}(p)$ and $Z_{0e}(p)$, respectively, viewed from the fault point $F$. The inductive line impedances are omitted in these diagrams. Their inclusion will modify the resultant curves but slightly, as they are small relative to other system impedances. Fundamental-frequency positive- and zero-sequence capacitive reactances $x_{c1}$ and $x_{c0}$, respectively, are divided by 3 to obtain their third harmonic values. $x_{1e}$ is multiplied by 3 to obtain its third harmonic value. The zero-sequence third harmonic reactance of the load, including load-end transformer, indicated by $3x_{0e}$, is taken equal to $3x_{1e}$. These assumptions in regard to the load reactance are arbitrary, as is also the assumption that the third harmonic zero-sequence reactance $3x_0$ viewed from $F$ toward the sending-end transformer is one-half $3x_2$, where $3x_2$ is the corresponding positive-sequence third harmonic reactance. Modifications in these arbitrary assumptions will modify the resonance regions indicated by the curves of Fig. 7 which are given merely as *approximate* resonance regions. For any given problem, more exact calculations, based on assumptions which apply to that particular problem, can be made.

**Problem 2.** A 60-cycle 10,000-kva salient-pole generator without amortisseur winding is connected through a transformer to a 110-kv, 30-mile, open-circuited transmission circuit. In per unit, based on generator rating, $x_d'' = x_d' = 0.30$; $x_q'' = x_q' = x_q = 0.70$; $r_1 = 0.01$; $r_2 = 0.03$; transformer impedance $= 0.01 + j0.085$. The transformer is connected Δ-Y, with the neutral of the Y on line side solidly grounded. The three-phase transmission circuit is 4/0, 19-strand hard-drawn copper;

$r = 0.276$ ohm per mile; $d = 0.528$ inch; $s = 15$ feet; earth resistivity $\rho = 100$ ohms (meter cube); average height of conductor above ground is 45 feet. There are no ground wires.

For what type or types of short circuit at $F$, the terminals of the transformer on the line side, will there be high third harmonic voltages?

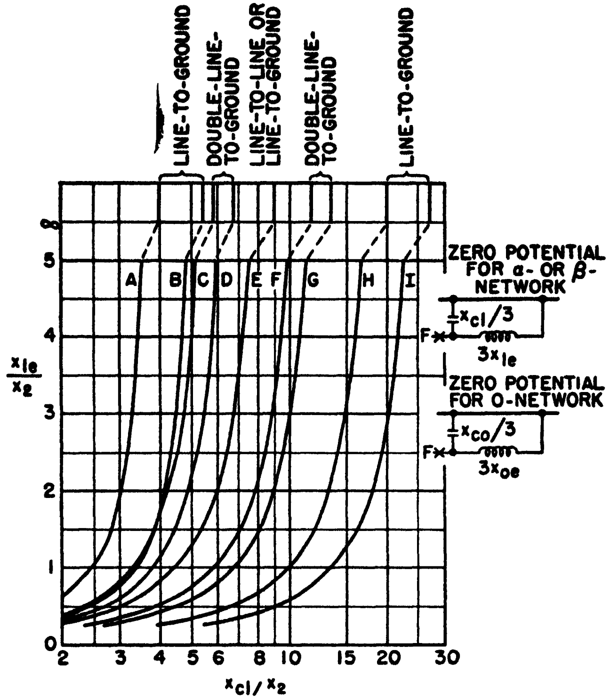

FIG. 7. Approximate third harmonic resonance regions for various types of faults. $A$, line-to-ground fault, grounded system $x_0 = x_2/2$, $x_{0e} = x_{1e}$, $C_0 = 0.6C_1$; $B$, line-to-ground fault, grounded system, $x_0 = x_2/2$, $x_{0e} = x_{1e}$, $C_0 = C_1$; $C$, double-line-to-ground fault, grounded system, $x_0 = x_2/2$, $x_{0e} = x_{1e}$, $C_0 = 0.6C_1$; $D$, double-line-to-ground fault, grounded system, $x_0 = x_2/2$, $x_{0e} = x_{1e}$, $C_0 = C_1$; $E$, line-to-line or line-to-ground fault; $F$, double-line-to-ground fault, ungrounded system, $C_0 = 0.6C_1$; $G$, double-line-to-ground fault, ungrounded system, $C_0 = C_1$; $H$, line-to-ground fault, ungrounded system, $C_0 = 0.6C_1$; $I$, line-to-ground fault, ungrounded system, $C_0 = C_1$.

***Solution.*** The 60-cycle transmission line constants are

$$Z_1 = 30(0.276 + j0.822) = 8.28 + j24.7 \text{ ohms}$$

$$Y_1 = 30(0 + j5.15)10^{-6} = 0 + j154.5 \times 10^{-6} \text{ mho}$$

$$Z_0 = 30(0.562 + j2.73) = 16.9 + j81.9 \text{ ohms}$$

$$Y_0 = 30(0 + j2.83)10^{-6} = 0 + j84.9 \times 10^{-6} \text{ mho}$$

The 180-cycle transmission line constant $Z_0$ is

$$Z_0 = 30(1.135 + j7.58) = 34.0 + j226.4 \text{ ohms}$$

Based on 10,000 kva and 110 kv in the transmission circuit, the 60-cycle per unit impedances are

$$
\begin{aligned}
Z_1 \text{ (gen + trans)} &= 0.02 \;\; + j0.385 \\
Z_2 \text{ (gen + trans)} &= 0.04 \;\; + j0.585 \\
Z_0 \text{ (trans)} &= 0.01 \;\; + j0.085 \\
\tfrac{1}{2} Z_1 \text{ (line)} &= 0.003 + j0.010 \\
\frac{1}{Y_1} \text{ (line)} &= 0 - j5.35 \\
\tfrac{1}{2} Z_0 \text{ (line)} &= 0.007 + j0.034 \\
\frac{1}{Y_0} \text{ (line)} &= 0 - j9.75
\end{aligned}
$$

With the line represented by its nominal $T$, at 60 cycles,

$$Z_{1e}\,(1) = \tfrac{1}{2} Z_1 \text{ (line)} + \frac{1}{Y_1} \text{ (line)} = 0.003 - j5.34$$

$$Z_{0e}\,(1) = \tfrac{1}{2} Z_0 \text{ (line)} + \frac{1}{Y_0} \text{ (line)} = 0.007 - j9.72$$

The 180-cycle per unit values are

$$
\begin{aligned}
Z_1 = Z_2 \text{ (gen + trans)} &= 0.04 \;\; + j1.755 \\
Z_0 \text{ (trans)} &= 0.01 \;\; + j0.255 \\
\tfrac{1}{2} Z_1 \text{ (line)} &= 0.003 + j0.030 \\
\tfrac{1}{2} Z_0 \text{ (line)} &= 0.014 + j0.094
\end{aligned}
$$

With the line represented by its nominal $T$, at 180 cycles,

$$Z_{1e}\,(3) = \tfrac{1}{2} Z_1 \text{ (line)} + \frac{1}{3Y_1} \text{ (line)} = 0.003 - j1.753$$

$$Z_{0e}\,(3) = \tfrac{1}{2} Z_0 \text{ (line)} + \frac{1}{3Y_0} \text{ (line)} = 0.014 - j3.156$$

When calculated values are substituted in the $\alpha$, $\beta$, and 0 networks of Fig. 4, it will be seen that a condition of near resonance exists in the unfaulted $\beta$ network. High third harmonic voltages are therefore to be expected with a line-to-ground fault. For a line-to-line fault, there will be near resonance in the $\alpha$ circuit of Fig. 5, with resultant high third harmonic voltage on the unfaulted phase. The actual magnitude of the resultant voltages will be limited by corona and transformer and generator losses and will not be as high as calculated.

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# CHAPTER IX

## SYSTEM PROTECTION — RELAYS

The successful operation of a power system is dependent upon the maintenance of adequate insulation for all live conducting parts. Despite all efforts on the part of system designers and operators to prevent a breakdown of insulation, failures at present do occur. For a power system to continue the supply of power to its consumers after a fault, it is essential that the section of the system containing the fault be isolated promptly from the remainder; or the system must be so arranged and operated that an insulation failure is not followed by a flow of short-circuit current. Although the latter procedure is possible in a few instances, through the use of Petersen coils (Volume I, page 176), in a great majority of cases it is necessary to disconnect the faulted section from the remainder of the system. It is the task of the protective relays and their associated circuit breakers to accomplish this. The chief function of system protection is to enable the continued rendering of service to consumers by segregating a faulted section as quickly as possible. Fortunately, a successful service-protective system also accomplishes the desirable result of minimizing damage to equipment at the point of fault or elsewhere; but, in general, the chief emphasis is upon the protection of service and not of apparatus.

To minimize the disturbance to the remainder of the system when a circuit is switched out of service, the area disconnected should be as small as possible. In other words, the minimum number of circuit breakers and only those nearest the fault should be opened. Because of the many and varied types of disturbances which may occur in a power system, and the large number of circuits of different characteristics which are controlled by relays, there are many types of relays. These are continually being improved or replaced by others better suited to maintaining continuity of service.

It is only because of differences between fault conditions and normal conditions that relays are able to detect the presence of a fault, locate it, and operate the breakers which disconnect it from the system. To insure protection against all types of faults, such as short circuits involving one or more conductors, open conductors, simultaneous short circuits, more than one system of relays may be used on circuits which are important in the maintenance of continuity of service. These systems of relays may be coordinated so that, if the main system does not operate to clear the fault, a back-up system will come into action.

## DETERMINATION OF RELAY PERFORMANCE

The three factors which determine relay performance are

1. System conditions, such as voltages and currents, at relay locations.
2. Intermediate devices, such as potential transformers and current transformers, which supply the system quantities to the relays in a form usable by them.
3. Characteristics and range of operation of the relays.

This chapter is concerned with the methods of calculating system conditions only. It will be assumed that both the relay characteristics and the degree of fidelity of the information given to the relays are known.

## CURRENT AND VOLTAGE RELAYS

The system quantities most often used by protective relays are currents, voltages, and combinations of current and voltage. Current relays may use line currents or components thereof, such as symmetrical components. Similarly, voltage relays may use line voltages or components thereof.

Methods of determining initial symmetrical rms phase currents and voltages and their positive-, negative-, and zero-sequence (or $\alpha\beta 0$) components of current and voltage at any system point are given in Volume I for all types of short circuits, for one and two open conductors, and for simultaneous faults of various types. By applying these methods, the required initial quantities at the relay can be determined in cases where d-c component, natural-frequency terms, and harmonics are unimportant. Subtransient reactances are used in such calculations to determine initial symmetrical rms quantities. When relay operation is delayed and there are no appreciable power swings, transient reactances replace subtransient in calculations.

## RELAYS RESPONSIVE TO BOTH CURRENT AND VOLTAGE

The response of relays energized by combinations of current and voltage (directional distance, for example) may be determined by study of current, voltage, and phase angle but, because three variables are involved, the determination of the response is complex. Although the absolute magnitude of current or voltage cannot be entirely ignored, the problem is greatly simplified by considering the ratio of voltage to current. The expression " impedance seen by the relay " has been widely used in the literature to signify the ratio at the relay of the voltage and current for which the relay is connected. If the ratio of voltage to current is used, the number of variables is reduced to two, thereby permitting plotting on a single impedance chart with resistance and reactance as abscissa and ordinate, respectively. On this chart, the characteristic curve of the limit of operation of the relay, and the system impedances seen by the relay under specified operating

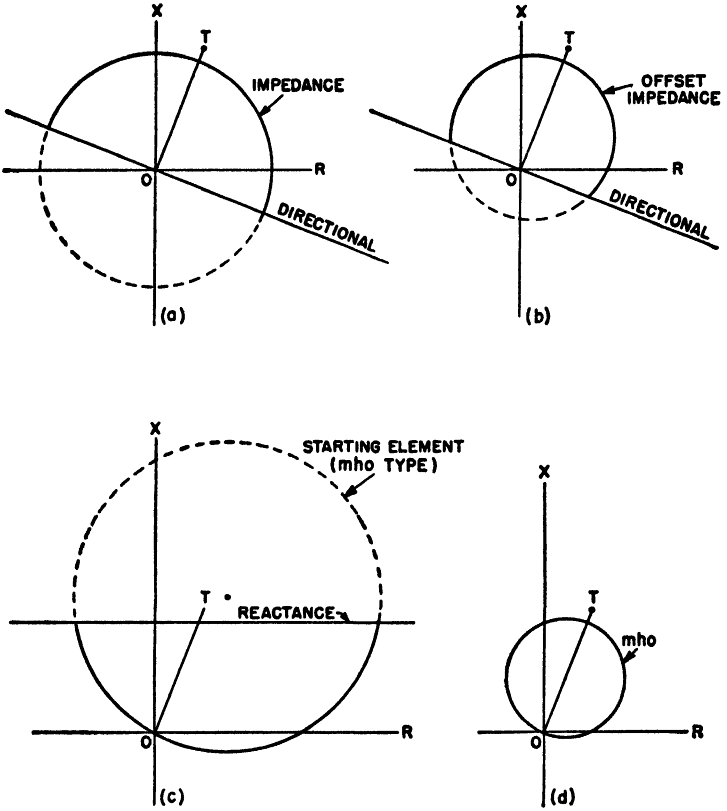


FIG. 1. Diagrams showing boundaries of tripping (within heavy lines) for (*a*) impedance relay with directional element; (*b*) offset impedance relay with directional element; (*c*) reactance relay with mho-type starting element; (*d*) mho relay.

conditions, are plotted to the same scale. If the system impedance seen by a relay falls within the boundary of the characteristic curve of the relay, the relay will operate; if it falls outside this boundary, the relay will not operate. This method of determining relay performance is given by J. H. Neher[1] in a paper on the performance of distance relays, in which he plots the characteristic curves of the boundary of operation of various types of relays in terms of resistance and reactance.

Figure 1 (parts *a*, *b*, *c*, and *d*) indicates characteristic curves of the limit of operation of four types of directional distance relays. All of them are set to protect approximately 90 per cent of the line between the relay location at the origin *O* and the terminal of the section of line at *T*. Either the impedance relay, Fig. 1(*a*), or the offset-impedance[2] relay, Fig. 1(*b*), will operate when the system impedance seen by it falls within its circular characteristic and is above and to the right of its directional element. The reactance-type relay, Fig. 1(*c*), will operate for system impedances which fall below its constant reactance straight line and within the circumference of its starting unit. The mho-type[3] relay, Fig. 1(*d*), will operate for system impedances which fall within the circumference of its circle which passes through the origin *O* with its diameter at an angle with the *x*-axis.

All the relays in Fig. 1 are adjustable as the diameters of the circles can be varied; adjustment is also possible of the location of the centers of the circles in Figs. 1(*b*), 1(*c*), and 1(*d*).

## POWER SWINGS

When a fault occurs, only those relays which control the faulty section should trip. During a fault, and also after a circuit is switched out of service, the synchronous machines of the system undergo changes in the relative positions of their rotors. With constant internal generated voltages, the phase angles between these voltages will vary until a new condition of equilibrium is established; or, if the disturbance is sufficiently severe, loss of synchronism will result. In cases where there will be no loss of synchronism, it is important that those relays which do *not* control the faulty section refuse to trip during power swings while the fault is on the system and also after it has been cleared.

### Equivalent Two-Machine System

For determining the performance of distance-type transmission-line relays during power swings, many three-phase power systems can be represented by two equivalent synchronous machines having balanced generated voltages $E_A$ and $E_B$ connected by a three-phase linear network. With such power systems, the boundary of operation of a relay and the system impedances seen by the relay during power swings can be expressed in the same terms and plotted on the same chart. Voltage and current connections of the relay, system characteristics, operating conditions, and relay location determine the system impedance seen by the relay; the restraining elements and the settings define the limit of operation of the relay. Consider a relay connected for specified voltage and current and whose boundary of operation is given in terms of resistance and reactance as in Fig. 1. All ratios of the specified voltage to the specified current seen by the relay during power swings can be indicated on an impedance chart and the boundary of operation of the relay superposed. When the system impedances seen by the relay fall within its boundary of operation, the relay will trip; if they fall outside, it will not trip.

**Assumptions.** The methods given later of deriving equations and constructing impedance charts apply to any three-phase system which can be replaced during power swings by two equivalent synchronous machines with balanced generated voltages $E_A$ and $E_B$, connected by a three-phase linear network with the relays located at any point $R$ in the network and a fault (if present) at any point $F$. In reducing a given system to one having two equivalent power sources, synchronous machines which can be represented by the same impedance in the direct and quadrature axes, and linear representation of loads, will be assumed. Initial transients, including d-c and natural-frequency components and the effects of saliency in synchronous machines during power swing will be neglected. Under these assumptions, there will be no harmonics because of saturation of other non-linear effects, or from unequal impedances in the two axes of rotating machines. Initial transients will be of short duration in the usual transmission system because of the relatively high ratio of resistance to reactance. $E_A$ and $E_B$ will be balanced sinusoidal voltages of fundamental frequency back of a single impedance in each of the two equivalent power sources, the phase angle between them varying during power swings. In the usual transient swings, transient impedance is assumed; but, for swings which are long relative to the field time constants, an impedance higher than transient impedance may be a better representation.

**Relay Connections.** Equations for the impedances seen by the three relays at any system location $R$ are expressed in terms of the voltages and currents for which the relays are connected. Phase voltages and currents in these equations are replaced by their components of voltage and current in the system of components chosen for system analysis. (For illustration see [21]–[27].)

**System Data.** System characteristics and operating conditions (including type of fault and its location) determine the components of voltage and current at the relay location $R$ in terms of the applied voltages $E_A$ and $E_B$ and system impedances and admittances.

**Equations for System Impedances Seen by Relays during Power Swings.** If the components of voltage and current at the relay, in terms of $E_A$ and $E_B$, are substituted in the equation determined by relay connections, an equation for the impedance seen by the relay during power swings is obtained.

**General Equation.** The general equation for the impedance $Z_R$ seen by a relay at any point $R$ in a two-machine three-phase system, with or without a fault on the system, may be written

$$Z_R = \frac{V_R}{I_R} = \frac{H_A E_A + H_B E_B}{G_A E_A - G_B E_B} \qquad [1]$$

(Note that [33] which is the ratio of [31] to [32] can be expressed in the form of [1].)

In [1], positive direction of current is from $A$ to $B$. $E_A$ and $E_B$ are balanced *internal* voltages of fundamental frequency in the two equivalent synchronous machines of the system. $H_A$, $H_B$, $G_A$, and $G_B$, under the stated assumptions, are constant coefficients which depend upon relay connections, system characteristics, operating conditions, and relay location. As the numerator in [1] represents a voltage and the denominator a current, $H_A$ and $H_B$ are complex numbers and $G_A$ and $G_B$ complex admittances.

*Special Case.* Equation [1] is applicable during power swings under symmetrical system conditions to a circuit or a section of the system between terminals $A$ and $B$ if $E_A$ and $E_B$ are balance *terminal* voltages of fundamental frequency at $A$ and $B$, respectively. In this case, the magnitudes of $E_A$ and $E_B$ as well as the phase angle between them will vary during power swings.

Equation [1] may be expressed in terms of $E_A/E_B$ or $E_B/E_A$, whichever is preferred.

$$Z_R = \frac{H_A}{G_A} - \frac{\left(\dfrac{H_A}{G_A} + \dfrac{H_B}{G_B}\right)}{1 - \dfrac{E_A}{E_B}\left(\dfrac{G_A}{G_B}\right)} = -\frac{H_B}{G_B} + \frac{\left(\dfrac{H_A}{G_A} + \dfrac{H_B}{G_B}\right)}{1 - \dfrac{E_B}{E_A}\left(\dfrac{G_B}{G_A}\right)} \qquad [2]$$

The equations in [2] are general. As given, they define the ratio of voltage to current. Similar equations, however, may be used to define the ratio of two currents or of two voltages, or any ratio which can be expressed in the general form of [1].

For any given relay location $R$ and operating conditions (including type and location of fault), $H_A/G_A$, and $H_B/G_B$ are constant impedances because they are ratios of complex numbers to complex admittances. $G_A/G_B$ (or $G_B/G_A$), being the ratio of two admittances, is a complex number.

Let

$$\frac{H_B}{G_B} = Z_x \tag{3a}$$

$$\frac{G_A}{G_B} = K \tag{3b}$$

$$\frac{H_A}{G_A} = Z_z \tag{3c}$$

$$\frac{H_A}{G_A} + \frac{H_B}{G_B} = Z_y \tag{3d}$$

If the above equations are substituted in [2], the impedance $Z_R$ seen by a relay at $R$ becomes

$$Z_R = Z_z - \frac{Z_y}{1 - \dfrac{E_A K}{E_B}} = -Z_x + \frac{Z_y}{1 - \dfrac{E_B}{E_A K}} \tag{4}$$

where $Z_x$, $Z_z$, and $Z_y = Z_x + Z_z$ are impedances, and $K$ in the general case is a complex number. In special cases, $Z_x$ or $Z_z$ may be zero, or $K$ may be unity. If positive direction for current is from $B$ to $A$, instead of from $A$ to $B$, the signs of the above equations for $Z_R$ will be reversed.

In terms of magnitudes and angle, let $Z_x = |Z_x|\angle\theta_x$, $Z_z = |Z_z|\angle\theta_z$, $Z_y = |Z_y|\angle\theta_y$, $K = |K|\angle\theta_k$, $E_A/E_B = |E_A/E_B|\angle\delta$, $E_B/E_A = |E_B/E_A|\angle{-\delta}$, where bars enclosing a quantity indicate its magnitude only, and $\delta$ is the angle by which $E_A$ leads $E_B$. With complex impedances and ratios in polar form, [4] becomes

$$\begin{aligned} Z_R &= |Z_z|\angle\theta_z - \frac{|Z_y|\angle\theta_y}{1 - \left|\dfrac{E_A K}{E_B}\right|\angle\delta + \theta_k} \\ &= -|Z_x|\angle\theta_x + \frac{|Z_y|\angle\theta_y}{1 - \left|\dfrac{E_B}{E_A K}\right|\angle{-\delta - \theta_k}} \end{aligned} \tag{4a}$$

Equations [4] and [4*a*] give $Z_R$ in two different forms which are not independent of each other. Either form can be obtained from the other, since $Z_y = Z_x + Z_z$.

## CONSTRUCTION OF SYSTEM IMPEDANCE CHARTS

System impedance charts, based on [4], can be constructed when $Z_x$, $Z_z$, $Z_y = Z_x + Z_z$, and $K$ have been determined. Two methods of obtaining these constants will be discussed later. For the present, let it be assumed that these constants are known or can be determined.

Figure 2 shows a system impedance chart constructed from known values of $Z_x$, $Z_z$, $Z_y$, and $K$. This chart will be used to illustrate the procedure in the construction of system impedance charts, which is the same with or without a fault on the system.

Two points on the impedance chart can be obtained by equating first $E_A$, and then $E_B$, to zero. From [4], with $E_A = 0$,

$$Z_R = Z_z - Z_y = -Z_x \qquad [5]$$

With $E_B = 0$,

$$Z_R = Z_z = -Z_x + Z_y \qquad [6]$$

In Fig. 2, $-Z_x$ locates point $A$, corresponding to $E_A = 0$; this point can also be located by $Z_z - Z_y$. $Z_z$ locates point $B$, corresponding to $E_B = 0$; this point can also be located by $-Z_x + Z_y$. The line $AB$ connecting points $A$ and $B$ and measured from $A$ to $B$ has the magnitude and slope of $Z_y = |Z_y|\underline{/\theta_y}$. The origin represents $R$ the location of the relay. Points $A$ and $B$ represent the terminals at which the voltages $E_A$ and $E_B$, referred to the relay, are applied. $E_A$ and $E_B$ are balanced internal generated voltages in the equivalent two-machine system, with or without a fault on the system, or *balanced* voltages at the terminals of a circuit under symmetrical system conditions.

Let

$$\phi = \delta + \theta_k \quad \text{or} \quad \delta = \phi - \theta_k \qquad [7]$$

where $\delta$ is the phase angle by which $E_A$ leads $E_B$, and $\theta_k$ is the angle of $K$ when written $K = |K|\underline{/\theta_k}$.

Substitution of [7] in [4*a*] gives

$$Z_R = |Z_z|\underline{/\theta_z} - \frac{|Z_y|\underline{/\theta_y}}{1 - \left|\dfrac{E_A K}{E_B}\right|\underline{/\phi}} = -|Z_x|\underline{/\theta_x} + \frac{|Z_y|\underline{/\theta_y}}{1 - \left|\dfrac{E_B}{E_A K}\right|\underline{/-\phi}} \qquad [8]$$

A third point $M$ in Fig. 2 can be obtained from [8], if $|E_A/E_B| = |1/K|$ and $\phi = \delta + \theta_k = 180°$. Then the denominator of either fraction in [8] is 2, and

$$Z_R = M = Z_z - \frac{Z_y}{2} = -Z_x + \frac{Z_y}{2} \qquad [9]$$

Either of the impedance equations in [9] locates $M$ at the midpoint of the line $AB$. This point is shown in Fig. 2.

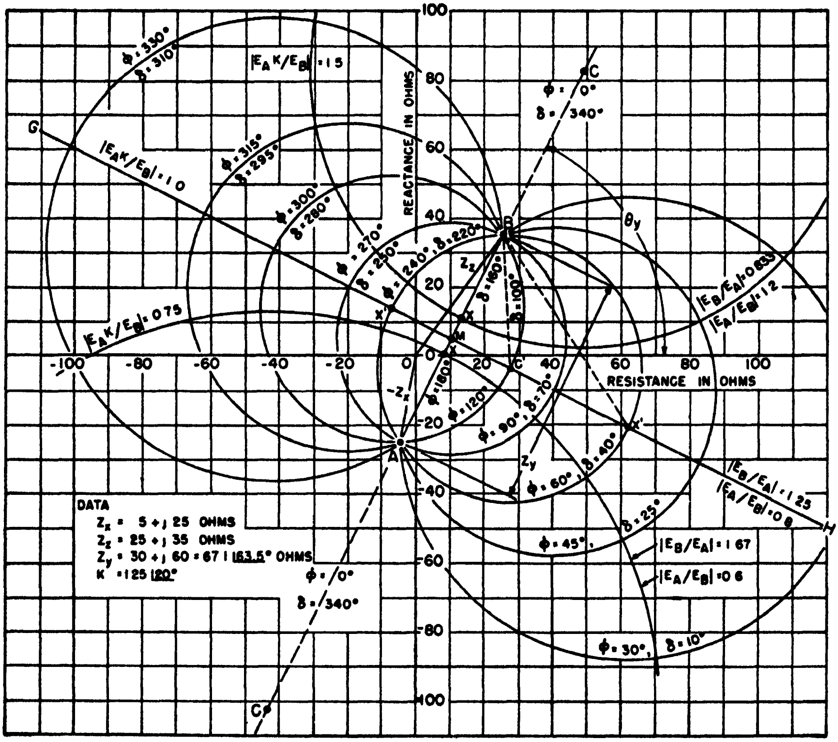


FIG. 2. System impedance chart in ohms.

For known constants $Z_x$, $Z_y$, $Z_z$, and $K$, there are two variables in [8]: the ratio $|E_A/E_B|$ and the angle $\delta$ in $\phi = \delta + \theta_k$.

**Circles for Constant Values of $|E_A/E_B|$.** For any constant value of $|E_A/E_B|$ and all values of the angle $\phi = \delta + \theta_k$ between 0° and 360°, the denominators of the fractions in [8] are equations of circles. It has been shown,[4] and proof is given in Appendix C, that the reciprocal of the equation of a circle is also the equation of a circle. The constant numerators of the fractions in [8] multiply centers and radii of the circles; the constant first term is added to the centers of the circles but it does not change their radii. If $r$ is the length of the radius and $C$ is the center of the circle referred to the origin and $Z_R$ is expressed in terms of $E_A/E_B$, from [C–9] and [C–10] of Appendix C, with $A = 1$ and $B = |E_AK/E_B|$,

$$r = \frac{|Z_y| \left|\frac{E_AK}{E_B}\right|}{\left|\left|\frac{E_AK}{E_B}\right|^2 - 1\right|} \qquad [10]$$

$$C = Z_z - \frac{Z_y}{1 - \left|\frac{E_AK}{E_B}\right|^2} = \left(Z_z - \frac{Z_y}{2}\right) + \frac{Z_y}{2} \, \frac{\left|\frac{E_AK}{E_B}\right|^2 + 1}{\left|\frac{E_AK}{E_B}\right|^2 - 1} \qquad [11]$$

The line of centers of the circles for constant values of $|E_A/E_B|$ is given by [11]. The term in parentheses, $Z_z - (Z_y/2)$, locates the midpoint of $AB$; the term following it locates $C$ along the line $AB$, or $AB$ extended. If $MC$ is the distance from $M$ to $C$ measured along the line of centers $AB$ (extended) in the direction $MB$,

$$MC = \left|\frac{Z_y}{2}\right| \frac{\left|\frac{E_AK}{E_B}\right|^2 + 1}{\left|\frac{E_AK}{E_B}\right|^2 - 1} \qquad [12]$$

If $|E_AK/E_B|$ is greater than unity, $MC$ in [12] will be positive and will be laid off in the direction $MB$. If $MC$ is negative, it will be laid off in the direction $MA$.

In Fig. 2, let $X$ be the intersection of the circumference of any circle of radius $r$ with $AB$, and $MX$ the distance measured from $M$ to $X$ in the positive direction $MB$. If the outer bars enclosing the denominator in [10] are removed, and [10] then subtracted from [12], the distance $MX$ is

$$MX = \left|\frac{Z_y}{2}\right| \frac{\left[\left|\frac{E_AK}{E_B}\right| - 1\right]^2}{\left|\frac{E_AK}{E_B}\right|^2 - 1} = \left|\frac{Z_y}{2}\right| \frac{\left|\frac{E_AK}{E_B}\right| - 1}{\left|\frac{E_AK}{E_B}\right| + 1} \qquad [13]$$

If $|E_AK/E_B|$ is greater than unity, $MX$ is positive and will be laid off along $AB$ in the direction $MB$. If $|E_AK/E_B|$ is less than unity, $MX$ will be negative and will be laid off in the direction $MA$. This is shown in Fig. 2 for $|E_AK/E_B| = 1.50$ and 0.75, respectively. $MX$

given by [13] and $MC$ given by [12] will always be measured in the same direction from $M$. When radii are long relative to the length $AB$, instead of using [11] or [12] to locate $C$, a higher degree of precision can usually be secured (if one is not an expert draftsman) by laying off $MX$ calculated from [13] and then adding $r$ given by [10] to find the center $C$ of the circle which cuts $AB$ at $X$.

Table I gives $r$ and $MX$ in terms of $|Z_y|$ corresponding to $|E_AK/E_B|$ where $r$ is the length of the radius and $MX$ is the distance from $M$, the midpoint of $AB$, to $X$, the intersection of the circle with $AB$.

In terms of $|E_B/E_A|$, the equations for $r$ and $MX$ are

$$r = \frac{|Z_y| \left| \dfrac{E_B}{E_AK} \right|}{\left| 1 - \left| \dfrac{E_B}{E_AK} \right|^2 \right|} \qquad [14]$$

$$MX = \left| \frac{Z_y}{2} \right| \frac{1 - \left| \dfrac{E_B}{E_AK} \right|}{1 + \left| \dfrac{E_B}{E_AK} \right|} \qquad [15]$$

If $|E_B/(E_AK)|$ is less than unity, $MX$ and $MC$ $(= |MX| + r)$ will be laid off along $AB$ (or $AB$ extended) in the positive direction $MB$. If $|E_B/(E_AK)|$ is greater than unity, $MX$ and $MC$ will be laid off in the negative direction $MA$. As $|E_B/(E_AK)|$ is the reciprocal of $|E_AK/E_B|$, circles for values of $|E_A/E_B|$ and $|E_B/E_A|$ which are reciprocals of each other will be identical. This is shown in Fig. 2 where the circles are labeled in terms of both $|E_A/E_B|$ and $|E_B/E_A|$.

If $|E_A/E_B| = |1/K|$, $|E_AK/E_B| = |E_B/(E_AK)| = 1$; and from [10] or [14], $r = \infty$. From [12], $MC = \infty$. The circle, in this case, becomes a straight line perpendicular to $AB$. As $MX$ from [13] or [15] is zero, the straight line passes through $M$ and is the perpendicular bisector of the line $AB$. This straight line $GMH$ is shown in Fig. 2.

**Arcs of Circles[5] for Constant Angles $\delta$ Between $E_A$ and $E_B$.** The second term in either of the equations in [8] can be equated to $1/D$ if numerator and denominator are divided by $|Z_y|\underline{/\theta_y}$. From the second equation of [8], with $-\phi$ replaced by $-\delta - \theta_k$,

$$D = \frac{1}{|Z_y|}\underline{/-\theta_y} - \left| \frac{E_B}{E_AKZ_y} \right| \underline{/-\delta - \theta_k - \theta_y} \qquad [16]$$

where $|Z_y|$, $\theta_y$, $|K|$, and $\theta_k$ are constants. For a constant value of $\delta$ but variable values of $|E_B/E_A|$ between 0 and $\infty$, $D$ in [16] is a straight

## TABLE I

### RADII AND INTERSECTIONS WITH $AB$ OF CIRCLES FOR CONSTANT VALUES OF $|E_A/E_B|$ IN TERMS OF $|Z_y| = AB$ AND $|E_AK/E_B|$

| $\lvert E_AK/E_B\rvert$ | $r^*/\lvert Z_y\rvert$ | $MX^*/\lvert Z_y\rvert$ | $\lvert E_AK/E_B\rvert$ | $r^*/\lvert Z_y\rvert$ | $MX^*/\lvert Z_y\rvert$ | $\lvert E_AK/E_B\rvert$ | $r^*/\lvert Z_y\rvert$ | $MX^*/\lvert Z_y\rvert$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | −0.5000 | 0.9 | 4.737 | −0.0263 | 1.55 | 1.105 | 0.1078 |
| 0.1 | 0.101 | −0.4091 | 0.95 | 9.744 | −0.0128 | 1.6 | 1.026 | 0.1154 |
| 0.2 | 0.208 | −0.3333 | 1.0 | ∞ | 0 | 1.65 | 0.958 | 0.1226 |
| 0.3 | 0.330 | −0.2692 | 1.05 | 10.244 | 0.0122 | 1.7 | 0.899 | 0.1296 |
| 0.4 | 0.476 | −0.2143 | 1.10 | 5.238 | 0.0238 | 1.75 | 0.848 | 0.1364 |
| 0.5 | 0.667 | −0.1667 | 1.15 | 3.566 | 0.0349 | 1.8 | 0.804 | 0.1429 |
| 0.55 | 0.789 | −0.1452 | 1.20 | 2.727 | 0.0455 | 1.85 | 0.764 | 0.1491 |
| 0.6 | 0.938 | −0.1250 | 1.25 | 2.222 | 0.0556 | 1.9 | 0.728 | 0.1552 |
| 0.65 | 1.126 | −0.1061 | 1.3 | 1.884 | 0.0652 | 1.95 | 0.696 | 0.1610 |
| 0.7 | 1.373 | −0.0882 | 1.35 | 1.641 | 0.0745 | 2.0 | 0.667 | 0.1667 |
| 0.75 | 1.714 | −0.0714 | 1.4 | 1.458 | 0.0833 | 2.5 | 0.476 | 0.2143 |
| 0.8 | 2.222 | −0.0556 | 1.45 | 1.315 | 0.0918 | 5.0 | 0.208 | 0.3333 |
| 0.85 | 3.063 | −0.0405 | 1.5 | 1.200 | 0.1000 | 10.0 | 0.101 | 0.4091 |
|  |  |  |  |  |  | ∞ | 0 | 0.5000 |

* $r$ is the radius of the circle; $X$ is its intersection with $AB$. See Fig. 2. $MX$ is the distance from $M$ the midpoint of $AB$ to $X$ in the positive direction $MB$. The center of the circle $C$ is on $AB$ (extended). $MC$ has the sign of $MX$. $|MC| = r + |MX|$.

line. This line passes through $|1/Z_y|\underline{/-\theta_y}$ and has a slope $180° - (\delta + \theta_k + \theta_y)$. It is shown* in Appendix C that, if the equation of an admittance is a straight line extending to infinity in *both* directions, its reciprocal is the equation of a circle which passes through the origin. In the special case of a straight line which passes through the origin, the reciprocal is a circle of infinite radius, i.e., a straight line.

As $|E_B/E_A|$ is always positive, the straight line representing the admittance $D$ for a constant value of $\delta$ extends to infinity in one direction only as $|E_B/E_A|$ varies from 0 to $\infty$. Its reciprocal $1/D$ is therefore the equation of an arc of a circle, not a complete circle. The first term in [8] changes the center of the arc but does not change its radius. The equations of [8], for any constant value of $\delta$ and ratios $|E_B/E_A|$ between 0 to $\infty$, are therefore the equations of arcs of circles.

It is shown in [5] and [6] that the impedance $Z_R$ is $-Z_x = Z_z - Z_y$ when $E_A = 0$, and $-Z_x + Z_y = Z_z$ when $E_B = 0$. These impedances are indicated in Fig. 2 by points $A$ and $B$, respectively. As these two values of $Z_R$ are independent of the angle $\delta$ by which $E_A$ leads $E_B$, the arcs for all values of $\delta$ must pass through the points $A$ and $B$. The centers of all arcs will therefore lie on the perpendicular bisector $GMH$ of the line $AB$. The line $GMH$ in Fig. 2 is the curve for $|E_AK/E_B| = |E_B/(E_AK)| = 1$; it is also the line of centers of the constant $\delta$ arcs.

The intersection of the constant $\delta$ arcs with their line of centers $GMH$ can be obtained if $|E_B/(E_AK)|$ in [8] is replaced by unity. Let $X'$ indicate the point of intersection; then $X'$ referred to the origin is given by the equation

$$\begin{aligned} X' &= -Z_x + \frac{Z_y}{1 - \underline{/-\phi}} = -Z_x + \frac{Z_y}{1 - \cos\phi + j\sin\phi} \\ &= -Z_x + \frac{Z_y(1 - \cos\phi - j\sin\phi)}{2(1 - \cos\phi)}. \\ &= -Z_x + \frac{Z_y}{2}\left(1 - j\frac{\sin\phi}{1 - \cos\phi}\right) \\ &= -Z_x + \frac{Z_y}{2} + \frac{|Z_y|\underline{/\theta_y - 90°}}{2\tan\frac{\phi}{2}} \end{aligned} \qquad [17]$$

---

* Equations [C-13] and [C-14] of Appendix C were not available to the author at the time reference 5 was written. The original development is given here to illustrate a method of procedure which may be useful in other connections. The student can develop [19] and [20] more simply by using [C-13] and [C-14].

The first two terms in [17] locate $M$ at the midpoint of $AB$. The third term gives the distance $MX'$, measured from $M$ to $X'$ along a line perpendicular to $Z_y$ at $M$, i.e., along the line $GMH$. The distance $MX'$ from $M$ to $X'$ measured along the line $GMH$ in the positive direction $MH$ is

$$MX' = \frac{|Z_y|}{2 \tan \dfrac{\phi}{2}} = |Z_y| \frac{\cot \dfrac{\phi}{2}}{2} \tag{18}$$

If cot $(\phi/2)$ is positive, $MX'$ is positive; if cot $(\phi/2)$ is negative, $MX'$ is negative. This is indicated in Fig. 2 where $MX'$ is positive for values of $\phi$ between 0° and 180°, and negative for values of $\phi$ between 180° and 360°. The intersections $X'$ are shown for $\phi = 60°$ and 240°.

From [18] and Fig. 2, the angles $AX'M$ and $BX'M$ are each equal to $\phi/2$. If $C'$ is the center of the arc, the angles $AC'M$ and $BC'M$ will each be equal to $\phi$ (an angle inscribed in a circle is one-half the angle at the center subtended by the same arc). The distance $MC'$ is therefore

$$MC' = \frac{|Z_y|}{2 \tan \phi} = \frac{|Z_y|}{2 \tan (\delta + \theta_k)} = |Z_y| \frac{\cot \phi}{2} \tag{19}$$

If cot $\phi$ is positive, the center $C'$ is in the direction $MH$; if cot $\phi$ is negative, $C'$ is in the direction $MG$. Values of $\delta$ or of $\phi = \delta + \theta_k$ which differ by 180° will have the same center $C'$; but, from [18], their intersection $X'$ with $GMH$ will be on opposite sides of $M$. The complete circle is divided into two arcs by points $A$ and $B$ which are common to all $\delta$ arcs. This is shown in Fig. 2, where the arcs for $\phi =$ 60° and $\phi =$ 240° have the same center $C'$ but their intersections $X'$ with $GMH$ are on opposite sides of $M$.

When the center $C'$ has been located, the radius $r'$ is equal to the distance from $C'$ to $A$ or $B$. The equation for the radius $r'$ is

$$r' = C'A = C'B = \frac{|Z_y|}{|2 \sin \phi|} = \frac{|Z_y|}{|2 \sin (\delta + \theta_k)|} \tag{20}$$

To construct charts for constant values of $|E_B/E_A|$ or $|E_A/E_B|$ and $\delta$, the assumed value of $|E_B/E_A|$ is divided by $K$ to obtain $|E_B/(E_AK)|$, or the assumed value of $|E_A/E_B|$ is multiplied by $|K|$ to obtain $|E_AK/E_B|$, and the assumed value of $\delta$ is added to $\theta_k$ to obtain $\phi = \delta + \theta_k$. Circles for constant values of $|E_B/(E_AK)|$ or $|E_AK/E_B|$ and arcs for constant values of $\phi$ are then constructed but circles and arcs are graduated in terms of $|E_B/E_A|$ or $|E_A/E_B|$ and $\delta$, respectively. This is illustrated in Fig. 2, where circles and arcs are labeled in terms of $|E_A/E_B|$, $|E_B/E_A|$ and $\delta$, as well as in terms of $|E_AK/E_B|$ and $\phi$.

Table II gives $MC'$, the distance of the center of the arc $C'$ from $M$, measured along $GMH$, the perpendicular bisector of $AB$, in terms of $\phi$ and $|Z_y|$; and the sign of $MX'$ is indicated. For values of $\phi$ not included in Table II, $MC'$ can be calculated from [19], and the sign of $MX'$ determined from [18].

TABLE II

LOCATION OF CENTERS OF ARCS FOR CONSTANT VALUES OF $\delta$ IN TERMS OF $|Z_y| = AB$ AND $\phi = \delta + \theta_k$

| $\phi° = \delta+\theta_k$ | $^*MC'/\|Z_y\|$ | $\phi° = \delta+\theta_k$ | $^*MC'/\|Z_y\|$ | $\phi° = \delta+\theta_k$ | $^*MC'/\|Z_y\|$ |
|---|---|---|---|---|---|
| 0, 180 | ∞ | 60, 240 | 0.2887 | 120, 300 | −0.2887 |
| 5, 185 | 5.715 | 65, 245 | 0.2332 | 125, 305 | −0.3501 |
| 10, 190 | 2.836 | 70, 250 | 0.1820 | 130, 310 | −0.4196 |
| 15, 195 | 1.866 | 75, 255 | 0.1340 | 135, 315 | −0.5000 |
| 20, 200 | 1.374 | 80, 260 | 0.0882 | 140, 320 | −0.596 |
| 25, 205 | 1.072 | 85, 265 | 0.0437 | 145, 325 | −0.714 |
| 30, 210 | 0.866 | 90, 270 | 0 | 150, 330 | −0.866 |
| 35, 215 | 0.714 | 95, 275 | −0.0437 | 155, 335 | −1.072 |
| 40, 220 | 0.596 | 100, 280 | −0.0882 | 160, 340 | −1.374 |
| 45, 225 | 0.500 | 105, 285 | −0.1340 | 165, 345 | −1.866 |
| 50, 230 | 0.4196 | 110, 290 | −0.1820 | 170, 350 | −2.836 |
| 55, 235 | 0.3501 | 115, 295 | −0.2332 | 175, 355 | −5.715 |

* $MC'$ is the distance from $M$ the midpoint of $AB$ to $C'$ the center of the arc on $GMH$. See Fig. 2. $MX'$ is the distance from $M$ to $X'$, the intersection of the arc with $GMH$ in the positive direction $MH$. If $\phi$ is between 0° and 180°, $MX'$ is positive; if $\phi$ is between 180° and 360°, $MX'$ is negative.

**General Impedance Chart.**[5] A general chart for determining the system impedances seen by relays during power swings was first suggested and constructed by A. J. McConnell.

Reference to [10]–[15] and [18]–[20] will show that when points $A$ and $B$ corresponding to $E_A = 0$ and $E_B = 0$, respectively, have been located, the set of circles for constant values of $|E_B/(E_AK)|$ or $|E_AK/E_B|$ and the arcs for constant values of $\phi = \delta + \theta_k$ can be determined from the length and slope of the line $AB$ which represents the impedance $Z_y = |Z_y|\underline{/\theta_y}$. If line $AB$ is taken as unity or 100 per cent, and a chart constructed with complete sets of circles and arcs of circles for constant values of $|E_B/(E_AK)|$ or $|E_AK/E_B|$ and $\phi = \delta + \theta_k$, respectively, this chart can be used for any system if

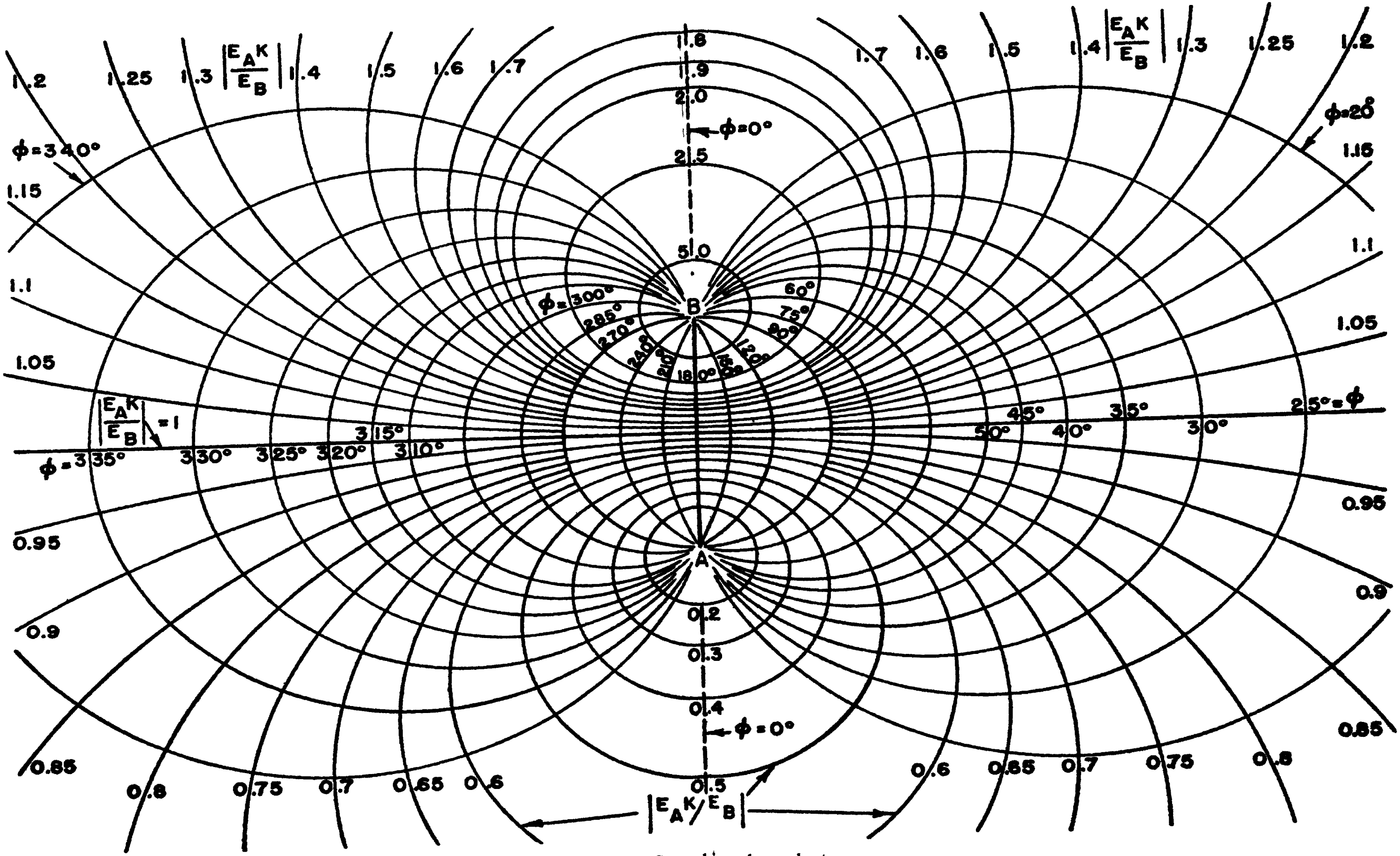


Fig. 3. General impedance chart.

impedances are expressed in per unit of a base impedance equal to $|Z_y| = AB$. The general chart is shown in Fig. 3 with line $AB$ in a vertical position. The length $AB = |Z_y|$ in this chart is 5 cm. Any convenient length may be selected. There are no coordinates on the chart. It consists of circles and arcs of circles and the straight lines $AB$ (extended) and $GMH$ perpendicular to $AB$ at its midpoint $M$. The circles are graduated in terms of $|E_AK/E_B|$, the arcs in terms of $\phi = \delta + \theta_k$.

**Use of General Impedance Chart.** After $-Z_x$, $Z_z$, $|Z_y|$, $|K|$, and $\theta_k$ have been calculated, $-Z_x$ and $Z_z$ are then expressed in per unit of base ohms, where $|Z_y|$ = base ohms = 1.0 in per unit. On a chart in rectangular coordinates, with resistance and reactance in per unit as abscissa and ordinate, respectively, in which unity length = $|Z_y|$ is equal to the length $AB$ on the general chart, the origin $O$ and points $A$ and $B$ are located. If the rectangular coordinate paper is thin, it can be placed over the general chart so that points $A$ and $B$ on the rectangular coordinate paper coincide with $A$ and $B$ of the general chart. The circles on the general chart, graduated in terms of $|E_AK/E_B|$ or $|E_B/(E_AK)|$, can be interpreted in terms of $|E_A/E_B|$ or $|E_B/E_A|$, by dividing or multiplying, respectively, the indicated value by $|K|$. The arcs can be interpreted in terms of $\delta$ by subtracting $\theta_k$ from the indicated value of $\phi$. The circles and arcs which are required can be traced on the rectangular coordinate paper. Illustrations of this use of the general impedance chart are given in Problems 1 and 2.

## CONSTANTS $Z_x$, $Z_z$, $Z_y$, AND $K$

The constants $Z_x$, $Z_z$, $Z_y$, and $K$ required in the construction of system impedance charts can be determined when relay equations and system conditions are known. The first step in the procedure is to write the three relay equations in terms of the voltages and currents for which the relays are connected. These equations are then expressed in terms of the components of the *system of components* to be used in system analysis. The given system is reduced to an equivalent two-machine system and the one-line diagrams of its required component networks drawn.

As there is a wide choice of the voltages and currents for which the relays may be connected, the procedure in writing relay equations will be illustrated for *assumed* relay connections.

Line-to-line voltages and their corresponding delta currents are advantageous because the system impedance viewed from the relay will be the same on three-phase, phase-to-phase, to two-phase-to-ground faults in a symmetrical three-phase linear system having equal positive- and negative-sequence impedances, provided the fault involves the phases to which the relay is connected.[6] And this is approximately true even when positive- and negative-sequence impedances are unequal.

When distance relays are used for protection in case of single-phase-to-ground faults, they may be supplied with line currents and their corresponding line-to-ground voltages.

**Relay Equations.** With the relays connected for line-to-line voltages and their corresponding delta currents, the three relay equations at $R$ are

$$Z_{cb} = \frac{V_{cb}}{I_b - I_c} = \frac{V_b - V_c}{I_b - I_c} \tag{21}$$

$$Z_{ac} = \frac{V_{ac}}{I_c - I_a} = \frac{V_c - V_a}{I_c - I_a} \tag{22}$$

$$Z_{ba} = \frac{V_{ba}}{I_a - I_b} = \frac{V_a - V_b}{I_a - I_b} \tag{23}$$

where $V$ and $I$ in the above equations are sinusoidal voltages and currents, respectively, of fundamental frequency at $R$; subscripts $a$, $b$, $c$ refer to phases $a$, $b$, $c$, respectively; and phase order is $abc$.

The three impedance equations given by [21]–[23] in terms of phase quantities are expressed in terms of their positive- and negative-sequence components and also in terms of their $\alpha$ and $\beta$ components (see Chapter I) by the equations

$$Z_{cb} = \frac{V_b - V_c}{I_b - I_c} = \frac{(V_{a1} - V_{a2})}{(I_{a1} - I_{a2})} = \frac{V_\beta}{I_\beta} \tag{24}$$

$$Z_{ac} = \frac{V_c - V_a}{I_c - I_a} = \frac{\sqrt{3}(V_{a1} + V_{a2}) - j(V_{a1} - V_{a2})}{\sqrt{3}(I_{a1} + I_{a2}) - j(I_{a1} - I_{a2})} = \frac{\sqrt{3}V_\alpha + V_\beta}{\sqrt{3}I_\alpha + I_\beta} \tag{25}$$

$$Z_{ba} = \frac{V_a - V_b}{I_a - I_b} = \frac{\sqrt{3}(V_{a1} + V_{a2}) + j(V_{a1} - V_{a2})}{\sqrt{3}(I_{a1} + I_{a2}) + j(I_{a1} - I_{a2})} = \frac{\sqrt{3}V_\alpha - V_\beta}{\sqrt{3}I_\alpha - I_\beta} \tag{26}$$

where components of voltage are line-to-neutral voltages and components of current are line currents.

In a *symmetrical system with no unsymmetrical fault* there are only positive-sequence currents and voltages. For this condition, [24]–[26] are represented by a single equation:

$$Z_R = Z_{cb} = Z_{ca} = Z_{bc} = \frac{V_{a1}}{I_{a1}} \tag{27}$$

Equation [27], developed for specified relay connections, is applicable to any relay connections where the impedance $Z_R$ seen by the three relays at $R$ under symmetrical operation is determined by the ratio of positive-sequence line-to-neutral voltage to positive-sequence line current at $R$. Relays connected for line current and their corresponding line-to-ground voltage come under this classification.

During *unsymmetrical faults*, [24]–[26] in terms of components of current and voltage are applicable for the relay connections assumed. For any other voltage and current connections, equations for the impedances seen by the three relays at $R$ can be expressed in terms of the components of voltage and current of the selected system of components. The choice of the system of components is optional.

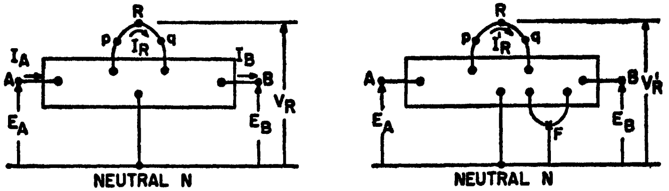
## System Impedance Diagrams

In the reduction of a three-phase power system to an equivalent two-machine system, the assumption is made that synchronous machines can be represented by the same impedance in their direct and quadrature axes. From this assumption, it follows that the negative-sequence impedances of the two equivalent synchronous machines may be assumed equal to their positive-sequence impedances. In the special case where $E_A$ and $E_B$ (hereinafter indicated by $V_A$ and $V_B$) are balanced voltages at the terminals $A$ and $B$ of a symmetrical three-phase circuit in a symmetrical system, the assumption of equal positive- and negative-sequence impedances is not required.

The assumption of a three-phase linear network with linear representation of loads between $E_A$ and $E_B$ does not exclude linear *unsymmetrical static circuits* in the two-machine system. Methods of treating such circuits are discussed briefly at the end of this chapter. Under the assumption of a *symmetrical* linear network between the two equivalent synchronous machines which have equal positive- and negative-sequence impedances, the negative-sequence impedance diagram of the system and also the $\alpha$ and $\beta$ impedance diagrams will be the same as the positive-sequence impedance diagram.

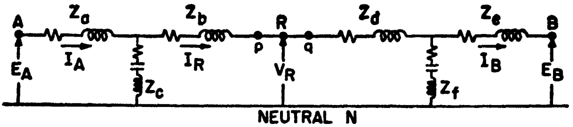
**Positive-Sequence Diagrams.** Three types of symmetrical three-phase linear networks between $E_A$ and $E_B$ will be considered: (1) a general network, (2) a network in which there is no connection between $E_A$ and $E_B$ except through the relays at $R$, (3) a single circuit of negligible capacitance. In reducing the positive-sequence diagram of a given system to one having two equivalent power sources, the identity is retained of the point $R$ at which the relays under consideration are located, and of the fault point $F$, if a fault is to be applied.

The positive-sequence diagram of a general symmetrical three-phase linear system with two power sources, or their equivalents, can be

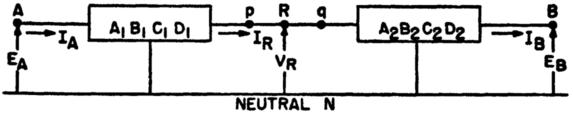


(a) - NO FAULT (b) - SOLID FAULT AT F

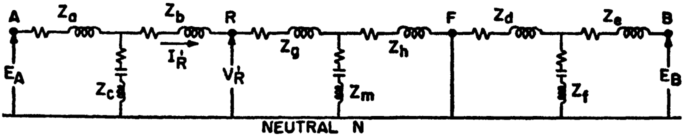
Fig. 4. Positive-sequence diagrams of a general symmetrical linear system between terminals $A$, $B$, and neutral $N$, represented as a box with section $pRq$ of the circuit at $R$ shown outside the box. In ($b$), the point of fault $F$, also shown outside the box, is shorted to neutral.



(a) - NO FAULT



(b) - NO FAULT



(c) - SOLID FAULT AT F

Fig. 5. Positive-sequence diagrams of a symmetrical linear system with no connection between $A$ and $B$ except through the relays at $R$. Represented in ($a$) by two three-terminal networks, in ($b$) by two sets of general circuit constants, and in ($c$) by three three-terminal networks with a fault to neutral at $F$.

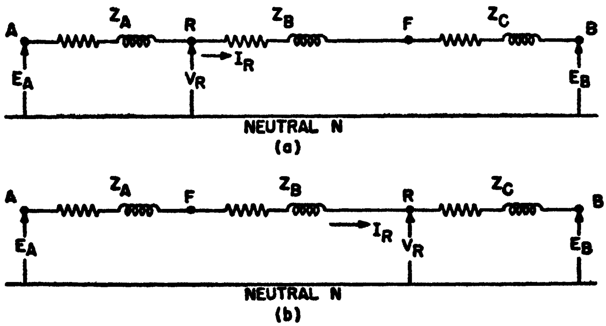


Fig. 6. Positive-sequence diagrams of a single circuit of negligible capacitance between $E_A$ and $E_B$. $R$ and $F$ are relay and fault locations, respectively.

represented as a single-phase network having terminals $A$ and $B$ at which voltages to neutral $E_A$ and $E_B$, respectively, are applied; when capacitance and constant impedance loads are present, there will be one or more terminals connected to neutral $N$. Figure 4(*a*) shows the positive-sequence diagram of a general network between $A$, $B$, and $N$ with no fault on the system. In Fig. 4(*b*), a solid fault is indicated at $F$. In these diagrams, the general network is represented as a box with three terminals, $A$, $B$, and $N$, brought outside the box; a portion $pRq$ of the circuit on which the relays are located is also shown outside the box. Points $p$ and $q$ are on opposite sides of the relays at $R$ but at negligible distances from $R$. $V_R$ is the voltage to neutral at $R$, and $I_R$ is the line current at $R$, positive direction of current being from $p$ to $q$, which is also from $A$ to $B$. In Fig. 4(*b*), a portion of the circuit at the fault $F$ is also shown outside the box.

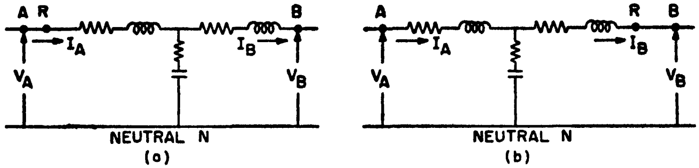


FIG. 7. Positive-sequence diagrams of a single symmetrical three-phase linear circuit between terminal voltages $V_A$ and $V_B$. (*a*) Relay location $R$ at $A$. (*b*) Relay location $R$ at $B$.

Figure 5(*a*) is the positive-sequence diagram of a system without a fault which can be separated into two parts which have no connection with each other except through the relays at $R$. The portions of the positive-sequence network on each side of $R$, shown as three-terminal networks, may contain constant impedance loads or line capacitance, or both. The impedance shunts $Z_c$ and $Z_f$ may be either inductive or capacitive; if there is neither capacitance nor constant impedance loads, they become infinite. In Fig. 5(*b*), the positive-sequence diagram of Fig. 5(*a*) is expressed in terms of two sets of general circuit constants. In Fig. 5(*c*), a solid fault is indicated at $F$. Figures 6(*a*) and 6(*b*) are special cases of Fig. 5(*a*) in which capacitances and constant impedance loads are negligible. In Fig. 6(*a*), the point $F$ at which a fault will be applied is between $R$ and $E_B$; in Fig. 6(*b*), $F$ is between $E_A$ and $R$.

Figures 7(*a*) and 7(*b*) are special cases of Fig. 5(*a*) in which $V_A$ and $V_B$ are terminal voltages (not internal generated voltages) at $A$ and $B$, the terminals of a symmetrical three-phase linear circuit of appreciable capacitance with relays located at $A$ and $B$, respectively. *In systems which cannot be replaced during power swings by two equivalent* *power sources*, system impedance charts can be constructed from Figs. 7(*a*) and 7(*b*) in terms of $|V_A/V_B|$ or $|V_B/V_A|$ and the phase angle $\delta$ by which $V_A$ leads $V_B$; but some auxiliary means, such as an a-c network analyzer, will be required to determine the range of $|V_A/V_B|$ during power swings under given conditions of operation.

### Methods of Evaluating $Z_x$, $Z_z$, $Z_y$, and $K$

Two methods will be given for evaluating the constants $Z_x$, $Z_z$, $Z_y = Z_x + Z_z$, and $K$ required in the construction of system impedance charts based on [4] or [4*a*]. Either method is applicable to any relay, connected for any specified voltage and current, located at any point $R$ in a three-phase power system with or without a fault at any point $F$ on the system, provided the assumptions previously given in regard to the system are applicable.

*First Method.* In this method, the components of current and voltage at the relays in terms of system impedances are derived for the given system operating under given conditions and substituted in the relay equations. The resulting equations give the impedances seen by the relays in terms of $E_A$, $E_B$, and system impedances. These equations are written in the form of [1], and $H_A$, $H_B$, $G_A$, and $G_B$ are evaluated by equating coefficients of $E_A$ and $E_B$ to corresponding coefficients in [1]. $Z_x$, $Z_z$, $Z_y$, and $K$ are then determined from [3*a*]–[3*d*].

*Second Method.* This method, proposed by E. W. Kimbark,[7] is based on the following considerations:

From [5] and [6], $-Z_x$ is the impedance seen by the relay when $E_A = 0$; and $Z_z$ is the impedance seen when $E_B = 0$, positive direction of current being from $E_A$ to $E_B$.

Let $Z_k$ = the impedance seen by the relay when $E_A = E_B$.

If $E_A = E_B$ in [4], $Z_R = Z_k$ and [4] becomes

$$Z_k = Z_z - \frac{Z_y}{1 - K} = -Z_x + \frac{Z_y}{1 - \dfrac{1}{K}}$$

Therefore,

$$K = \frac{Z_k + Z_x}{Z_k - Z_z} \qquad [28]$$

When $Z_k = \infty$, as is the case when capacitance is negligible and there is no unsymmetrical fault, $K = 1.0\underline{/0^\circ}$.

By this method, the three impedances $Z_x$, $Z_z$, and $Z_k$ are defined in terms of one impressed voltage each ($E_A$, $E_B$, and $E_A = E_B$), and can therefore be determined separately for the given relay connections under specified conditions.

### Power Swings during Symmetrical System Conditions

This case includes power swings after a fault has been cleared and initial transients have disappeared, and also power swings during symmetrical three-phase faults. In either case, there are only positive-sequence voltages and currents to be considered. With any relay connections for which the three relays at $R$ see the same impedance $Z_R$, expressed as the ratio of positive-sequence line-to-neutral voltage to positive-sequence line current, the following discussion applies.

**Determination of $Z_x$, $Z_z$, and $K$ with No Fault.** When an a-c network analyzer is used, the *second method* described above can be readily applied. The positive-sequence network between the two equivalent power sources at $A$ and $B$ is set up. A voltage to neutral $E_B$ is applied at $B$ with $A$ shorted to neutral, and voltages to neutral and line currents at all relay locations read. The ratio of voltage to current in the direction of applied voltage to any relay location $R$ will give $Z_x$ at $R$. Similarly, a voltage $E_A$ is applied at $A$ with $B$ shorted to neutral and $Z_z$ determined. Points $A$ and $B$ are then connected and a single voltage $E$ applied between this common point and neutral, and voltages to neutral and line currents in the direction $AB$ (when positive direction of current is from $A$ to $B$) read at all relay locations. The ratio of voltage to current at any relay location $R$ will give $Z_k$ at $R$. $K$ at $R$ is then calculated from [28], and $Z_x$ and $Z_z$ added to give $Z_y$.

When an a-c network analyzer is not used, either of the methods can be applied analytically. The use of both methods will be illustrated.

Equations for $Z_x$, $Z_y$, $Z_z$, and $K$ are given in Table III in terms of the transfer and driving-point impedances of the positive sequence diagrams shown in Figs. 4 to 7. The derivation of these equations follows a discussion of the transfer and driving-point impedances used in Table III.

### Definitions of Transfer and Driving-Point Impedances

In a single-phase network, such as a positive-sequence network, with voltages to neutral $N$ applied at terminals $A$ and $B$, and $R$ any point in the network, the following definitions apply:

$Z_{RR}$ = driving-point impedance between terminals $R$ and $N$

= ratio of the voltage to neutral applied at $R$ to the total current entering the network at $R$ when terminals $A$ and $B$ are shorted to neutral $N$

$Z_{RR-A}$, $Z_{RR-B}$ = components of $Z_{RR}$ in the directions $Rp$ and $Rq$ (also $RA$ and $RB$), respectively, see Fig. 4($a$)

= ratios of the voltage to neutral applied at $R$ to the components of current entering the network at $R$ and flowing through $p$ and $q$ (toward $A$ and $B$), respectively, when $A$ and $B$ are shorted to neutral

$Z_{AR} = Z_{RA}$ = transfer impedance between terminals $A$ and $N$ and terminals $R$ and $N$

= ratio of the voltage to neutral applied at $A$ to the total current leaving the network at $R$ when $R$ and $B$ are shorted to neutral

= ratio of the voltage to neutral applied at $R$ to the current leaving the network at $A$ when $A$ and $B$ are shorted to neutral

$Z_{BR} = Z_{RB}$ = transfer impedance between terminals $B$ and $N$ and terminals $R$ and $N$ (defined as $Z_{AR} = Z_{RA}$ above, with $B$ and $A$ interchanged)

$z_{AR} = z_{RA}$ = transfer impedance between terminals $A$ and $N$ and terminals $p$ and $q$ when $p$ and $q$ are connected through $R$

= ratio of the voltage to neutral applied at $A$ to the line current at $R$ when $B$ is shorted to neutral and $p$ and $q$ are connected through $R$ ($R$ is not shorted to neutral)

= ratio of the series voltage inserted at $R$ between $p$ and $q$ to the current at $A$ when $A$ and $B$ are shorted to neutral

$z_{BR} = z_{RB}$ = transfer impedance between terminals $B$ and $N$ and terminals $q$ and $p$ when $q$ and $p$ are connected through $R$ (defined as $z_{AR} = z_{RA}$ above, with $B$ and $A$ interchanged)

From the above definitions of $Z_{RR-A}$, $Z_{RR-B}$, and $Z_{RR}$, it follows that

$$Z_{RR} = \frac{Z_{RR-A}\,Z_{RR-B}}{Z_{RR-A} + Z_{RR-B}} \qquad [29]$$

**Transfer and Driving-Point Impedance from Impedance Diagrams.** The following equations give the transfer and driving-point impedances required in case 2, Table III, in terms of the branch impedances indicated in Fig. 5(*a*) and also in terms of the general circuit constants (Volume I, page 170) indicated in Fig. 5(*b*).

$$Z_{RR-A} = Z_b + \frac{Z_aZ_c}{Z_a + Z_c} = \frac{Z_aZ_b + Z_aZ_c + Z_cZ_b}{Z_a + Z_c} = \frac{B_1}{A_1} \qquad [30a]$$

$$Z_{RR-B} = Z_d + \frac{Z_eZ_f}{Z_e + Z_f} = \frac{Z_dZ_e + Z_dZ_f + Z_eZ_f}{Z_e + Z_f} = \frac{B_2}{D_2} \qquad [30b]$$

$$Z_{RA} = Z_{AR} = Z_a + Z_b + \frac{Z_aZ_b}{Z_c} = \frac{Z_aZ_b + Z_aZ_c + Z_bZ_c}{Z_c} = B_1 \qquad [30c]$$

$$Z_{RB} = Z_{BR} = Z_d + Z_e + \frac{Z_dZ_e}{Z_f} = \frac{Z_dZ_e + Z_dZ_f + Z_eZ_f}{Z_f} = B_2 \qquad [30d]$$

The circuit constants $A_1$ and $D_2$ in terms of the branch impedances in Fig. 5(*a*) and in terms of driving-point and transfer impedances are

$$A_1 = \frac{Z_a + Z_c}{Z_c} = \frac{Z_{AR}}{Z_{RR-A}} \qquad [30e]$$

$$D_2 = \frac{Z_e + Z_f}{Z_f} = \frac{Z_{BR}}{Z_{RR-B}} \qquad [30f]$$

The circuit constants $B_1$ and $B_2$ are the same as the transfer impedances $Z_{AR}$ and $Z_{BR}$, respectively.

**Derivation of Equations in Table III.** Equations for $Z_x$, $Z_z$, $Z_y$, and $K$ in terms of transfer and driving-point impedances in Figs. 4 to 7 will be derived by the first method previously described. Equations for the positive-sequence components of voltage and current at the relay, required for this method, are useful when power swings during unsymmetrical faults are to be considered.

*General Network between $E_A$ and $E_B$. See Fig. 4(a).* With $E_A$ and $E_B$ applied at $A$ and $B$, respectively, no current enters or leaves the network at $R$. Let $V_R$ be the resultant voltage to neutral at $R$. Under the principle of superposition, $R$ may be considered a network terminal at which a voltage to neutral $V_R$ is applied through zero impedance so that the current sent into the network at $R$ by $V_R$ just equals the sum of the currents sent out of the network at $R$ by $E_A$ and $E_B$: $V_R/Z_{RR}$ must equal the sum of $E_A/Z_{AR}$ and $E_B/Z_{BR}$ or

$$V_R = Z_{RR}\left(\frac{E_A}{Z_{AR}} + \frac{E_B}{Z_{BR}}\right) = \frac{Z_{RR-A}\,Z_{RR-B}}{Z_{RR-A} + Z_{RR-B}}\left(\frac{E_A}{Z_{AR}} + \frac{E_B}{Z_{BR}}\right) \qquad [31]$$

## TABLE III

### VALUES OF $Z_x$, $Z_z$, $Z_y$, AND $K$, IN TERMS OF DRIVING-POINT* AND TRANSFER IMPEDANCES* IN THE POSITIVE-SEQUENCE NETWORKS OF FIGS. 4–7 — NO FAULT — RELAY IMPEDANCES DETERMINED BY $V_{a1}/I_{a1}$

| Case | | $Z_x$ | $Z_z$ | $Z_y$ | $K$ |
|---|---|---|---|---|---|
| 1 | General network of Fig. 4(*a*). | $Z_{RR}\dfrac{z_{BR}}{Z_{BR}}$ | $Z_{RR}\dfrac{z_{AR}}{Z_{AR}}$ | $Z_{RR}\left(\dfrac{z_{AR}}{Z_{AR}}+\dfrac{z_{BR}}{Z_{BR}}\right)$ | $\dfrac{z_{BR}}{z_{AR}}$ |
| 2 | Connection between *A* and *B* only through relay. Fig. 5(*a*). | $Z_{RR-A}$ | $Z_{RR-B}$ | $Z_{RR-A}+Z_{RR-B}$ | $\dfrac{Z_{RR-A}Z_{BR}}{Z_{AR}Z_{RR-B}}$ |
| 3 | Similar to case 2 except negligible capacitance. Fig. 6(*a*). $Z_{RA} = Z_{RR-A}$; $Z_{RB} = Z_{RR-B}$. | $Z_{RR-A}$ | $Z_{RR-B}$ | $Z_{RR-A}+Z_{RR-B}$ | $1\underline{/0^\circ}$ |
| 4 | Relay located where $E_A$ is measured. Fig. 7(*a*). $Z_{RR-A} = 0$; $Z_{RR-A}/Z_{AR} = 1$. | 0 | $Z_{RR-B}$ | $Z_{RR-B}$ | $\dfrac{Z_{BR}}{Z_{RR-B}}$ |
| 5 | Relay located where $E_B$ is measured. Fig. 7(*b*). $Z_{RR-B} = 0$; $Z_{BR}/Z_{RR-B} = 1$. | $Z_{RR-A}$ | 0 | $Z_{RR-A}$ | $\dfrac{Z_{RR-A}}{Z_{AR}}$ |

* All the impedances in Table III can be obtained from the positive-sequence diagrams of Figs. 4–7 with *A* and *B* shorted to neutral and no fault on the system. With a voltage $E_R$ applied between neutral and *R*, and *A* and *B* shorted to neutral, let *I* be the current entering the network at *R* and $I_p$ and $I_q$ the components of *I* at *p* and *q*, respectively, which are also the components of *I* at *R* flowing toward *A* and *B*, respectively; let $I_A$ and $I_B$ be the currents leaving the network at *A* and *B*, respectively. Then $Z_{RR} = E_R/I$; $Z_{RR-A} = E_R/I_p$; $Z_{RR-B} = E_R/I_q$; $Z_{AR} = Z_{RA} = E_R/I_A$; $Z_{BR} = Z_{RB} = E_R/I_B$. The impedances $z_{AR}$ and $z_{BR}$ are required only for the general network. With a series voltage $e_R$ inserted at *R* in Fig. 4(*a*), and *A* and *B* shorted to neutral, let $I_A$ and $I_B$ be the currents at *A* and *B*, respectively, in the direction of the applied voltage. Then $z_{AR} = z_{RA} = e_R/I_A$; $z_{BR} = z_{RB} = e_R/I_B$.

The line current $I_R$ at $R$ in the direction $pRq$, with $R$ isolated from neutral, may be determined by superposing the line currents resulting from $E_A$ and $E_B$, separately applied. Thus,

$$I_R = \frac{E_A}{z_{AR}} - \frac{E_B}{z_{BR}} \qquad [32]$$

where positive direction of current is from $A$ to $B$.

The impedance $Z_R$ seen by the relay at $R$ in the general network of Fig. 4($a$) is obtained by dividing $V_R$ by $I_R$:

$$Z_R = \frac{V_R}{I_R} = \frac{Z_{RR}z_{AR}z_{BR}}{Z_{AR}Z_{BR}}\left(\frac{E_AZ_{BR} + E_BZ_{AR}}{E_Az_{BR} - E_Bz_{AR}}\right) \qquad [33]$$

If the coefficients of $E_A$ and $E_B$ in [33] are equated to corresponding coefficients in [1], $Z_x$, $Z_z$, $Z_y$, and $K$ can be calculated from [3$a$]–[3$d$]; or [33] can be rewritten in the form of [4], as in [33$a$] below.

$$Z_R = Z_{RR}\frac{z_{AR}}{Z_{AR}} - \frac{Z_{RR}\left(\dfrac{z_{AR}}{Z_{AR}} + \dfrac{z_{BR}}{Z_{BR}}\right)}{1 - \dfrac{E_A}{E_B}\left(\dfrac{z_{BR}}{z_{AR}}\right)} = -Z_{RR}\frac{z_{BR}}{Z_{BR}} + \frac{Z_{RR}\left(\dfrac{z_{AR}}{Z_{AR}} + \dfrac{z_{BR}}{Z_{BR}}\right)}{1 - \dfrac{E_B}{E_A}\left(\dfrac{z_{AR}}{z_{BR}}\right)} \qquad [33a]$$

Equation [33$a$] can be replaced by [4] if

$$Z_x = Z_{RR}\frac{z_{BR}}{Z_{BR}}$$

$$Z_z = Z_{RR}\frac{z_{AR}}{Z_{AR}}$$

$$Z_y = Z_{RR}\left(\frac{z_{AR}}{Z_{AR}} + \frac{z_{BR}}{Z_{BR}}\right)$$

$$K = \frac{z_{BR}}{z_{AR}}$$

These equations for $Z_x$, $Z_y$, $Z_z$, and $K$ are tabulated in Table III, case 1.

Equations [31] and [32] are general equations for voltage and current at $R$ in the positive-sequence network under any operating condition, provided the impedances involved are determined for that condition.

For special networks, where the only path between $E_A$ and $E_B$ is through the relays at $R$, $I_R$ is more easily calculated if the impedances $z_{AR}$ and $z_{BR}$ are replaced by more readily determined transfer and driving-point impedances.

*Special Network between $E_A$ and $E_B$. See Fig. 5(a).* When point $R$, the location of relays, divides the system into two parts which have no connection except through $R$, the line current $I_R$ at $R$ is given by either of the following equations.

$$I_R = \frac{E_A}{Z_{AR}} - \frac{V_R}{Z_{RR-A}}$$
$$I_R = \frac{V_R}{Z_{RR-B}} - \frac{E_B}{Z_{BR}} \qquad [34]$$

$V_R$ is given by [31].

Substitution of $V_R$ from [31] in either equation of [34] gives

$$\begin{aligned} I_R &= Z_{RR}\left(\frac{E_A}{Z_{AR}Z_{RR-B}} - \frac{E_B}{Z_{BR}Z_{RR-A}}\right) \\ &= \frac{Z_{RR-A}Z_{RR-B}}{Z_{RR-A} + Z_{RR-B}}\left(\frac{E_A}{Z_{AR}Z_{RR-B}} - \frac{E_B}{Z_{BR}Z_{RR-A}}\right) \end{aligned} \qquad [35]$$

$$Z_R = \frac{V_R}{I_R} = \frac{Z_{RR-A}Z_{RR-B}(Z_{BR}E_A + Z_{AR}E_B)}{Z_{RR-A}Z_{BR}E_A - Z_{RR-B}Z_{AR}E_B} \qquad [36]$$

$Z_x$, $Z_z$, $Z_y$, and $K$ for this case can be obtained from [36], as illustrated for case 1. They are tabulated in Table III, case 2. Or, by an alternate procedure, $z_{AR}$ and $z_{BR}$ can be evaluated for the special networks and substituted in case 1 of Table III to obtain equations for the other cases of Table III.

*Evaluation of $z_{AR}$ and $z_{BR}$ when the Only Path between $E_A$ and $E_B$ Is through the Relays at $R$.* If the coefficients of $E_A$ and $E_B$ in [35] are equated to the corresponding coefficients in [32], the following equations are obtained:

$$z_{AR} = \frac{Z_{AR}Z_{RR-B}}{Z_{RR}} \qquad [37a]$$

$$z_{BR} = \frac{Z_{BR}Z_{RR-A}}{Z_{RR}} \qquad [37b]$$

Equations [37*a*] and [37*b*] do not apply to the general network shown in Fig. 4(*a*).

**General Circuit Constants.** If the positive-sequence diagram is given in terms of general circuit constants, as in Fig. 5(*b*), the following equations apply:

$$E_A = V_R A_1 + I_R B_1 \qquad \text{and} \qquad E_B = V_R D_2 - I_R B_2$$

Solution of the above equations for $V_R$ and $I_R$ gives

$$V_R = \frac{E_A B_2 + E_B B_1}{A_1 B_2 + D_2 B_1} \qquad \text{and} \qquad I_R = \frac{E_A D_2 - E_B A_1}{A_1 B_2 + D_2 B_1}$$

The impedance $Z_R$ seen by a relay at $R$ is

$$Z_R = \frac{V_R}{I_R} = \frac{E_A B_2 + E_B B_1}{E_A D_2 - E_B A_1} \qquad [38]$$

If the coefficients of $E_A$ and $E_B$ in [38] are equated to corresponding coefficients in [1], and $Z_x$, $Z_z$, $Z_y$, and $K$ calculated from [3*a*]–[3*d*], the following equations are obtained:

$$Z_x = \frac{B_1}{A_1} = Z_{RR-A} \qquad [39a]$$

$$Z_z = \frac{B_2}{D_2} = Z_{RR-B} \qquad [39b]$$

$$Z_y = \frac{B_1}{A_1} + \frac{B_2}{D_2} = Z_{RR-A} + Z_{RR-B} \qquad [39c]$$

$$K = \frac{D_2}{A_1} = \frac{Z_{RR-A} Z_{BR}}{Z_{RR-B} Z_{AR}} \qquad [39d]$$

See [30*a*]–[30*f*] for relations between circuit constants and transfer and driving-point impedances.

The network of Fig. 5(*a*) will be used to illustrate the ***second method*** of determining $Z_x$, $Z_z$, and $K$ analytically.

Let positive direction of current be from $A$ to $B$. With a voltage $E_B$ applied at $B$ and $A$ shorted to neutral,

$$Z_x = \frac{V_R}{-I_R} = Z_{RR-A} \qquad [40]$$

Similarly, with $B$ shorted to neutral, and $E_A$ applied at $A$,

$$Z_z = \frac{V_R}{I_R} = Z_{RR-B} \qquad [41]$$

With equal voltages $E_A = E_B = E$ applied at $A$ and $B$, $Z_k = Z_R$ given by [36] if $E_A$ and $E_B$ are replaced by $E$:

$$Z_k = \frac{Z_{RR-A} Z_{RR-B}\ (Z_{AR} + Z_{BR})}{Z_{RR-A} Z_{BR} - Z_{RR-B} Z_{AR}} \qquad [42]$$

Substitution of the above value of $Z_k$ in [28] gives

$$K = \frac{Z_{RR-A}Z_{BR}}{Z_{RR-B}Z_{AR}} \tag{43}$$

Equations [39*a*]–[39*d*], and [40], [41], and [43] agree with the corresponding equations of case 2, Table III.

*Single Circuit between $E_A$ and $E_B$ of Negligible Capacitance. See Fig.* 6(*a*). If there are no impedance loads and capacitance is negligible, the positive-sequence diagram of Fig. 5(*a*) becomes that of Fig. 6(*a*). The voltage $V_R$ is given by [31] and the current $I_R$ by [34].

With *no fault* on the system,

$$Z_{AR} = Z_{RR-A} \qquad \text{and} \qquad Z_{BR} = Z_{RR-B}$$

The equations of case 3, Table III, are determined from those of case 2 by replacing $Z_{AR}$ and $Z_{BR}$ by $Z_{RR-A}$ and $Z_{RR-B}$, respectively.

*Relays at Terminals A and B.* Let the relays be located at the terminals $A$ and $B$ of a general three-phase circuit, and let $V_A$ and $V_B$ be the positive-sequence *terminal* voltages to neutral at the relays. Let $I_A$ and $I_B$ be the positive-sequence line currents at $A$ and $B$, respectively, positive direction of line current being from $A$ to $B$. The positive-sequence diagram of this system is shown as a three-terminal network between $A$, $B$, and neutral $N$ in Figs. 7(*a*) and 7(*b*), with the relay location $R$ at $A$ in Fig. 7(*a*), and at $B$ in Fig. 7(*b*).

Let $R$ in Fig. 7(*a*) approach $A$; then, in the limit, $Z_{AR} = Z_{RR-A} = 0$, and $Z_{AR}/Z_{RR-A} = 1$. Let $R$ in Fig. 7(*b*) approach $B$; then, in the limit, $Z_{BR} = Z_{RR-B} = 0$; and $Z_{BR}/Z_{RR-B} = 1$. By substitution of these values in the equations of case 2, Table III, the equations of cases 4 and 5, respectively, are obtained.

When the single circuit between $V_A$ and $V_B$ in Figs. 7(*a*) and 7(*b*) is defined in terms of general circuit constants $A$, $B$, $C$, and $D$, the constants $Z_x$, $Z_z$, $Z_y$, and $K$ can be expressed in terms of the circuit constants. From Fig. 7(*a*), with the relay at $A$,

$$V_B = DV_A - BI_A \tag{44}$$

$$I_A = \frac{DV_A - V_B}{B} \tag{45}$$

$$Z_R = \frac{V_A}{I_A} = \frac{BV_A}{DV_A - V_B} \tag{46}$$

The coefficients of $V_A$ and $V_B$ in [46] can be equated to those of $E_A$ and $E_B$, respectively, in [1], and $Z_x$, $Z_y$, $Z_z$, and $K$ then calculated from [3a]–[3d]:

$$Z_x = 0; \qquad Z_z = Z_y = \frac{B}{D}; \qquad K = D \tag{47}$$

From Fig. 7(*b*), with the relay at $B$,

$$V_A = AV_B + BI_B$$

$$Z_R = \frac{V_B}{I_B} = \frac{BV_B}{V_A - AV_B}$$

$$Z_x = Z_y = \frac{B}{A}; \qquad Z_z = 0; \qquad K = \frac{1}{A} \tag{48}$$

## Power Swings during Symmetrical Three-Phase Faults

With a symmetrical three-phase fault at any location which does not separate the applied voltages $E_A$ and $E_B$, the procedure in determining $Z_x$, $Z_z$, and $K$ is the same as without a fault except that the fault point $F$ in the positive-sequence network is shorted to neutral. See Fig. 4(*b*). When $E_A$ and $E_B$ are separated by the three-phase fault, there will be no power surges after initial transients have disappeared. In Figs. 5(*c*), 6(*a*), or 6(*b*), where a three-phase short circuit separates $E_A$ and $E_B$, the impedance seen by the relays at $R$ is a constant impedance. With a fault at $F$ in Fig. 5(*c*), $Z_R = Z_g + Z_m Z_h/(Z_m + Z_h)$. With a fault at $F$ in Fig. 6(*a*), $Z_R = Z_B$. With a fault at $F$ in Fig. 6(*b*), $Z_R = -Z_B$, positive direction of current being from $A$ to $B$.

## Power Swings during Unsymmetrical Faults

In power swings during unsymmetrical faults, the system impedances seen by the three relays at any location $R$ will be different. In the general case, three impedance charts, one for each relay, will be required. After $Z_x$, $Z_y$, $Z_z$, and $K$ have been determined for each relay, system impedance charts are constructed as in Fig. 2, or the general chart of Fig. 3 can be used as directed.

**Zero-Sequence Network.** When zero-sequence currents and voltages do not appear in the relay equations, they need not be calculated; but zero-sequence impedances will be required for ground faults or open conductors.

Let $Z_0$ be the zero-sequence impedance viewed from the fault at $F$. Let $z_0$ be the impedance offered to a *series voltage* inserted in the zero-sequence network at $F$. If $F$ is on a single *ungrounded* circuit, $Z_0$ and $z_0$ will both be infinite; if the circuit is grounded at one terminal only, $Z_0$ will be finite but $z_0$ infinite. If $F$ is on one of two parallel circuits, bussed at both ends, grounded or ungrounded, $z_0$ will be finite.

**Designation of Phases.** With phase $a$ as reference phase, let the fault be at any system point $F$; and let the phases be designated so that line-to-line and double-line-to-ground short circuits involve phases $b$ and $c$ and line-to-ground short circuits occur on phase $a$. In the case of one open conductor, let the opening occur at $F$ in phase $a$; with two open conductors, let the openings occur at $F$ in phases $b$ and $c$.

**Solution by Means of an A-C Network Analyzer.** When an a-c network analyzer is used, the networks of the arbitrarily chosen system of components are set up and the conditions imposed by the fault satisfied. The components of voltage and current at the relay location $R$ which appear in the three relay equations are obtained under given fault conditions, first with any voltage $E_B$ applied at $B$ and $E_A$ equated to zero. If positive direction for current is from $A$ to $B$, substitution of these components in the three relay equations will give three values of $-Z_x$. The three values of $Z_z$ are similarly obtained with $E_A$ applied at $A$ and $E_B$ equated to zero, and the three values of $Z_k$ with equal voltages $E$ applied at $A$ and $B$. The three values of $K$ are then calculated from [28]. It is important in determining $Z_k$ and $K$ that components of current be measured in the direction assumed as positive.

**Analytical Solution.** When positive- and negative-sequence impedances are equal, unsymmetrical system conditions are usually more readily analyzed by means of positive-plus-negative, positive-minus-negative and zero-sequence components (Volume I, Chapter V) or their related $\alpha\beta 0$ components (Volume I, Chapter X) than by the method of symmetrical components. When there is but a single dissymmetry, such as an unsymmetrical short circuit or one or two open conductors, in an otherwise symmetrical system, there is no connection between the positive-plus-negative and the positive-minus-negative networks. This makes it possible to use the positive-sequence network to represent both of these component networks, as explained in detail in Volume I, Chapter V.

## Conditions Imposed by Faults

**1. Line-to-Line Short Circuit at $F$ between Phases $b$ and $c$.** The positive-plus-negative network is unaffected by fault; point $F$ in the positive-minus-negative network is shorted directly to neutral.

2. **Line-to-Ground Short Circuit at *F* on Phase *a*.** Point $F$ in the positive-plus-negative network is shorted to neutral through $Z_0/2$; the positive-minus-negative network is unaffected.

3. **Double Line-to-Ground Short Circuit at *F* on Phases *b* and *c*.** Point $F$ in the positive-plus-negative network is shorted to neutral through $2Z_0$; point $F$ in the positive-minus-negative network is shorted directly to neutral.

4. **Phase *a* Open at *F*.** The impedance $2z_0$ is inserted in the positive-plus-negative network at $F$; the positive-minus-negative network is unaffected.

5. **Phases *b* and *c* Open at *F*.** The impedance $z_0/2$ is inserted in the positive-plus-negative network at $F$; the positive-minus-negative network is opened at $F$.

As the positive-plus-negative and the positive-minus-negative networks are both represented by the positive-sequence network, components of voltage and current in these networks can be evaluated from the equations for $V_R$ and $I_R$ already derived, *provided the transfer and driving-point impedances in these equations are determined under the conditions imposed by the fault.* $V_R$ is given by [31]; $I_R$ is given by [32] for the general network, and by [35] for a network in which the only path between $E_A$ and $E_B$ is through the relays at $R$. If zero-sequence components appear in the relay equations, they can be determined from the interconnection of the zero-sequence network with the positive-plus-negative network under the conditions imposed by the fault. After the components of voltage and current at the relay location $R$ are substituted in the three equations for the relays, these equations are then written in the form of [1] from which the values of $Z_x$, $Z_z$, $Z_y$, and $K$ are obtained, as previously described.

**Line-to-Line Fault.** To illustrate the procedure, a line-to-line fault at $F$ in Fig. 6($a$) will be assumed. The positive-plus-negative network and the positive-minus-negative network are both represented by Fig. 6($a$), where $Z_S = Z_A + Z_B + Z_C$.

As the positive-plus-negative network is unaffected by the fault, [31] and [35] are directly applicable. In these equations, $Z_{RR-A} = Z_{RA} = Z_A$; $Z_{RR-B} = Z_{RB} = Z_S - Z_A$. Therefore, from [31] and [35],

$$V_{a1} + V_{a2} = \frac{E_A(Z_S - Z_A) + E_B Z_A}{Z_S} \qquad [49]$$

$$I_{a1} + I_{a2} = \frac{E_A - E_B}{Z_S} \qquad [50]$$

With point $F$ in the positive-minus-negative network shorted to neutral, [31] and [35] in terms of transfer and driving-point impedances at $R$ still apply, but these impedances are determined with point $F$ shorted to neutral in Fig. 6($a$). Under this condition, $Z_{RR-A} = Z_{RA} = Z_A$, as above; but $Z_{RR-B} = Z_B$ and $Z_{RB} = \infty$. Equations [31] and [35] become

$$(V_{a1} - V_{a2}) = \frac{E_A Z_B}{Z_A + Z_B} \tag{51}$$

$$(I_{a1} - I_{a2}) = \frac{E_A}{Z_A + Z_B} \tag{52}$$

Equations [49]–[52], to be substituted in the relay equations, are independent of relay connections. To determine the impedances seen by the relays, the relay connections must be known and the relay equations written.

If the relays are connected for *line-to-line voltages and their corresponding delta currents*, the impedances seen by the three relays are determined by substitution of [49]–[52] in [24]–[26]. This gives

$$Z_{cb} = \frac{(V_{a1} - V_{a2})}{(I_{a1} - I_{a2})} = Z_B \tag{53}$$

$$\begin{aligned} Z_{ac} &= \frac{\sqrt{3}(V_{a1} + V_{a2}) - j(V_{a1} - V_{a2})}{\sqrt{3}(I_{a1} + I_{a2}) - j(I_{a1} - I_{a2})} \\ &= \frac{E_A\left[\dfrac{\sqrt{3}(Z_S - Z_A)}{Z_S} - j\dfrac{Z_B}{Z_A + Z_B}\right] + E_B\left[\dfrac{\sqrt{3}Z_A}{Z_S}\right]}{E_A\left[\dfrac{\sqrt{3}}{Z_S} - \dfrac{j}{Z_A + Z_B}\right] - E_B\left[\dfrac{\sqrt{3}}{Z_S}\right]} \\ &= \frac{E_A[\sqrt{3}(Z_B+Z_C)(Z_A+Z_B) - jZ_B Z_S] + E_B[\sqrt{3}Z_A(Z_A+Z_B)]}{E_A[\sqrt{3}(Z_A+Z_B) - jZ_S] - E_B[\sqrt{3}(Z_A+Z_B)]} \end{aligned} \tag{54}$$

If the coefficients of $E_A$ and $E_B$ in [54] are equated to the corresponding coefficients in [1], $Z_x$, $Z_z$, $Z_y$, and $K$, calculated from [3$a$]–[3$d$], for substitution in [4], have the following values:

$$Z_x = Z_A \tag{55}$$

$$K = 1 - j\frac{Z_S}{\sqrt{3}(Z_A + Z_B)} \tag{56}$$

$$Z_z = Z_B + \frac{\sqrt{3}Z_C(Z_A + Z_B)}{\sqrt{3}(Z_A + Z_B) - jZ_S} = Z_B + \frac{Z_C}{K} \tag{57}$$

$$Z_y = Z_x + Z_z = Z_A + Z_B + \frac{Z_C}{K} = \frac{Z_S}{K}\left(1 - j\frac{1}{\sqrt{3}}\right) \tag{58}$$

where $Z_S = Z_A + Z_B + Z_C$.

## TABLE IV

### VALUES OF $Z_x$, $Z_z$, $Z_y$, AND $K$ — LINE-TO-LINE FAULT BETWEEN PHASES $b$ AND $c$ AT $F$ IN FIG. 6($a$) OR 6($b$) AND RELAYS CONNECTED FOR LINE-TO-LINE VOLTAGES AND THEIR CORRESPONDING DELTA CURRENTS — POSITIVE DIRECTION OF CURRENT FROM $A$ TO $B$

| Figure | | $Z_x$ | $Z_z$ | $Z_y = Z_x - Z_z$ | $K$ |
|---|---|---|---|---|---|
| 6($a$) | $Z_{cb} = Z_B$ | | | | |
| | $Z_{ac}$ | $Z_A$ | $Z_B + \frac{Z_C}{K}$ | $Z_A + Z_B + \frac{Z_C}{K} = \frac{Z_S}{K}\left(1 - j\frac{1}{\sqrt{3}}\right)$ | $1 - j\frac{Z_S}{\sqrt{3}\,(Z_A + Z_B)}$ |
| | $Z_{ba}$ | $Z_A$ | $Z_B + \frac{Z_C}{K}$ | $Z_A + Z_B + \frac{Z_C}{K} = \frac{Z_S}{K}\left(1 + j\frac{1}{\sqrt{3}}\right)$ | $1 + j\frac{Z_S}{\sqrt{3}\,(Z_A + Z_B)}$ |
| 6($b$) | $Z_{cb} = -Z_B$ | | | | |
| | $Z_{ac}$ | $KZ_A + Z_B$ | $Z_C$ | $KZ_A + Z_B + Z_C = KZ_S\left(1 - j\frac{1}{\sqrt{3}}\right)$ | $\frac{1}{\left[1 - j\frac{Z_S}{\sqrt{3}\,(Z_B + Z_C)}\right]}$ |
| | $Z_{bc}$ | $KZ_A + Z_B$ | $Z_C$ | $KZ_A + Z_B + Z_C = KZ_S\left(1 + j\frac{1}{\sqrt{3}}\right)$ | $\frac{1}{\left[1 + j\frac{Z_S}{\sqrt{3}\,(Z_B + Z_C)}\right]}$ |

The equations for $Z_x$, $K$, $Z_z$, and $Z_y$ which appear in the equation for $Z_{ba}$ are similar to those given by [55]–[58] except that the *signs of all j-terms are reversed.*

Table IV gives equations for $Z_x$, $Z_z$, $Z_y$, and $K$ for three relays at $R$ in terms of the impedances $Z_A$, $Z_B$, $Z_C$, and $Z_S = Z_A + Z_B + Z_C$ of Figs. 6(*a*) and 6(*b*), with a line-to-line fault at $F$ between phases $b$ and $c$ and relays connected for line-to-line voltages and their corresponding delta currents. Note that the positions of $R$ and $F$ relative to $E_A$ and $E_B$ in Fig. 6(*a*) are reversed in Fig. 6(*b*).

**Systems with Unsymmetrical Static Circuits.** The methods of constructing system impedance charts, given here, apply also to three-phase linear systems with unsymmetrical static circuits, with or without an unsymmetrical fault on the system. The unsymmetrical circuit can be replaced by equivalent circuits in the $\alpha$, $\beta$, and 0 networks. In the case of a general unsymmetrical circuit in an otherwise symmetrical system, the $\alpha$, $\beta$, and 0 networks are mutually coupled (Volume I, Chapter X). The procedure in determining the required constants $Z_x$, $Z_z$, $Z_y$, and $K$ is too lengthy for discussion here. These constants, however, can be determined, and the work required is not excessive if an a-c network analyzer is available.

There are many unsymmetrical circuits which have equal impedances in two of the phases; if, in addition, the zero-sequence impedance of the circuit is infinite, as is the case with an open-delta transformer bank of two identical single-phase units, there will be no mutual coupling between the $\alpha$, $\beta$, and 0 networks. For this special case, the procedure in determining $Z_x$, $Z_z$, $Z_y$, and $K$ is similar to that for symmetrical three-phase systems except that $V_\alpha$ and $I_\alpha$ at the relay location $R$ are determined from the $\alpha$ diagram and $V_\beta$ and $I_\beta$ from the $\beta$ diagram, instead of using the positive-sequence diagram first for the $\alpha$ network and then for the $\beta$ network. Even without an unsymmetrical fault, the impedance seen by the relays in the three phases at $R$ will not be identical, and three equations will be required in all cases to determine the impedances seen by the three relays.

**Problem 1.** The relays are connected for line-to-line voltages and their corresponding delta currents. The given system can be represented as a single circuit of negligible capacitance between two equivalent generated voltages $E_A$ and $E_B$ with the relays located at $R$, as indicated in Fig. 6(*a*). A line-to-line fault between phases $b$ and $c$ occurs at $F$. The impedances $Z_A$, $Z_B$, and $Z_C$ shown in Fig. 6(*a*) are $Z_A = 20\underline{/80^\circ}$ ohms; $Z_B = 50\underline{/80^\circ}$ ohms; $Z_C = 30\underline{/80^\circ}$ ohms; $Z_S = 100\underline{/80^\circ}$ ohms. (It is not required that these impedance angles be equal; calculations are but slightly simplified by this assumption.) (*a*) Determine $Z_{cb}$, and construct charts for determining $Z_{ac}$ and $Z_{ba}$ during power swings. (*b*) Assuming $|E_A/E_B| = 1$, compare the impedances $Z_{ac}$ and $Z_{bc}$ determined under (*a*) with the impedances seen by the relays in the three phases at $R$ during power swings when there is no fault on the system.

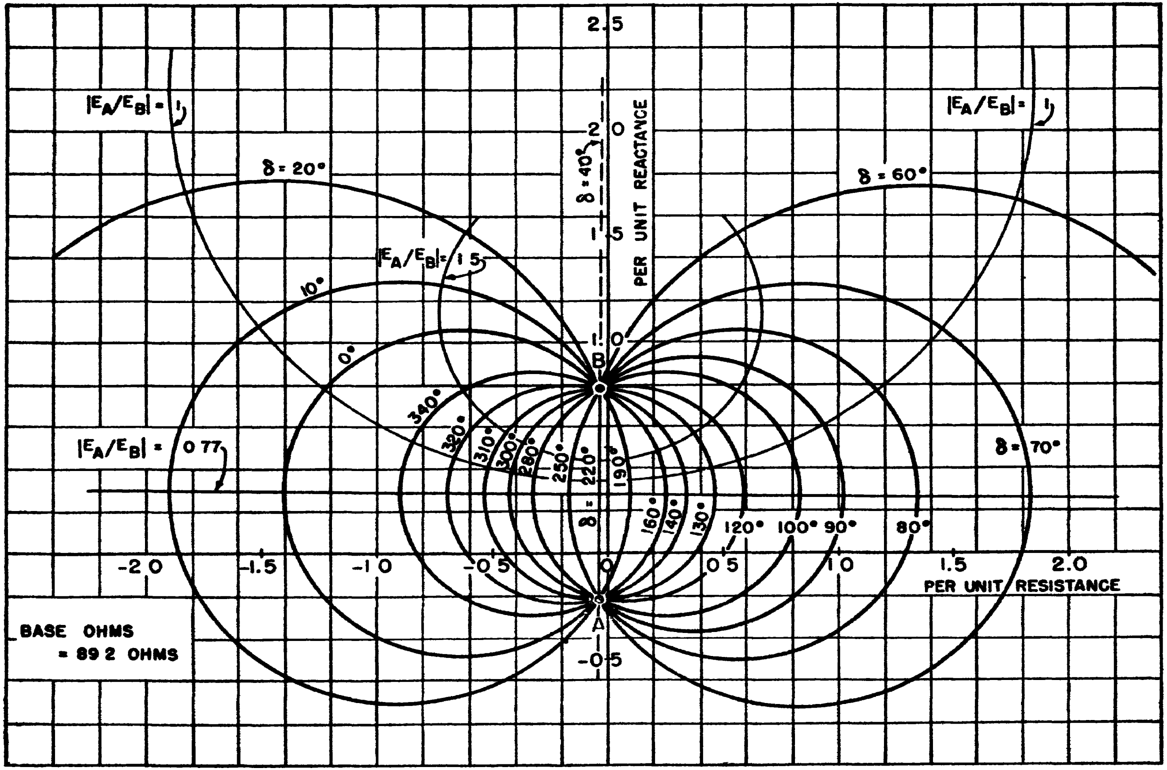


FIG. 8(*a*). Impedance chart for Problem 1. $Z_{ac}$ in per unit of 89.2 ohms $|E_A/E_B| = 1$.

*Solution* (*a*). From Table IV,

$$Z_{cb} = Z_B = 50\underline{/80^\circ} \text{ ohms}$$

*Chart for $Z_{ac}$.* From Table IV,

$$Z_x = Z_A = 20\underline{/80^\circ} \text{ ohms}$$

$$K = 1 - j\,\frac{Z_S}{\sqrt{3}(Z_A + Z_B)} = 1 - j\,\frac{100}{70\sqrt{3}} = 1 - j0.8248$$

$$= 1.296\underline{/-39.52^\circ} = 1.30\underline{/-40^\circ}, \text{ approximately}$$

$$Z_y = \frac{Z_S}{K}\left(1 - j\,\frac{1}{\sqrt{3}}\right) = \frac{100\underline{/80^\circ}\ (1.155\underline{/-30^\circ})}{1.296\underline{/-39.52^\circ}} = 89.2\underline{/89.52^\circ} \text{ ohms}$$

The chart of the impedance $Z_{ac}$ will be expressed in per unit of a base impedance equal to $|Z_y| = 89.2$ ohms. In per unit of 89.2 ohms, $Z_y = 1.00\underline{/89.52^\circ} = 0.009 + j1.00$; $Z_x = 0.2244\underline{/80^\circ} = 0.039 + j0.221$, $Z_z = Z_y - Z_x = -0.030 + j0.779$. $K$, being a complex ratio, remains unchanged. $|K| = 1.30$ and $\theta_k = -40^\circ$. The chart of $Z_{ac}$ given in Fig. 8(*a*) was constructed by following the steps given in the construction of Fig. 2, except that impedances are in per unit of $|Z_y| = 1.0$. The origin $O$ is located, and a convenient per unit scale chosen. $-Z_x = -0.039 - j0.221$ and $Z_z = -0.030 + 0.779$ are plotted, locating points $A$ and $B$, respectively. The line $AB$ is drawn, its midpoint $M$ is determined, the perpendicular $GMH$ to $AB$ at $M$ is erected, and $AB$ is extended in both directions.

Values of $|E_A/E_B|$ or $|E_B/E_A|$ covering the operating voltage range are selected and $|E_AK/E_B| = 1.30\ |E_A/E_B|$ calculated. The lengths of the radii $r$ and the distances $MX$ from $M$, the midpoint of $AB$, to the intersections $X$ of the circles for constant values $|E_AK/E_B|$ with the line $AB$ are read from Table I. For example, if $|E_A/E_B| = 1$, $|E_AK/E_B| = 1.30$; and from Table I, with $|Z_y| = 1$, $r = 1.884$ and $MX = 0.065$. In Fig. 8(*a*) the circle for $|E_B/E_A| = 1$ is shown. The straight line $GMH$ which corresponds to $|E_AK/E_B| = 1$ is also shown. This straight line corresponds to $|E_A/E_B| = 1/|K| = 0.77$.

With $\theta_k = -40^\circ$, $\phi = \delta + \theta_k = \delta - 40^\circ$; the distance $MC'$ of the center of the arcs for any constant values of $\delta$ can be read from Table II corresponding to $\phi = \delta - 40^\circ$. Thus: for $\delta = 0^\circ$, $\phi = 0 - 40^\circ = 320^\circ$; for $\delta = 60^\circ$, $\phi = 20^\circ$. The line $AB$ corresponding to $\phi = 180^\circ$ is the curve for $\delta = \phi - \theta_k = 220^\circ$; the line $AB$ (extended) is the curve for $\delta = 0 - \theta_k = 40^\circ$. The arcs for $\phi = 90^\circ$ and $270^\circ$ (circle with $AB$ as diameter) are arcs for $\delta = 130^\circ$ and $310^\circ$, respectively. These and other values of $\delta$ are indicated in Fig. 8(*a*).

*Chart for $Z_{ba}$.* From Table IV,

$$Z_x = Z_A = 20\underline{/80^\circ}$$

$$K = 1 + j\,\frac{Z_S}{\sqrt{3}(Z_A + Z_B)} = 1 + j0.8248 = 1.296\underline{/39.52^\circ}$$

$$|K| = 1.30$$

$$\theta_k = 40^\circ, \text{ approximately}$$

$$Z_y = \frac{Z_S}{K}\left(1 + j\,\frac{1}{\sqrt{3}}\right) = \frac{100\underline{/80^\circ}(1.155\underline{/30^\circ})}{1.296\underline{/39.52^\circ}} = 89.2\underline{/70.48^\circ} \text{ ohms}$$

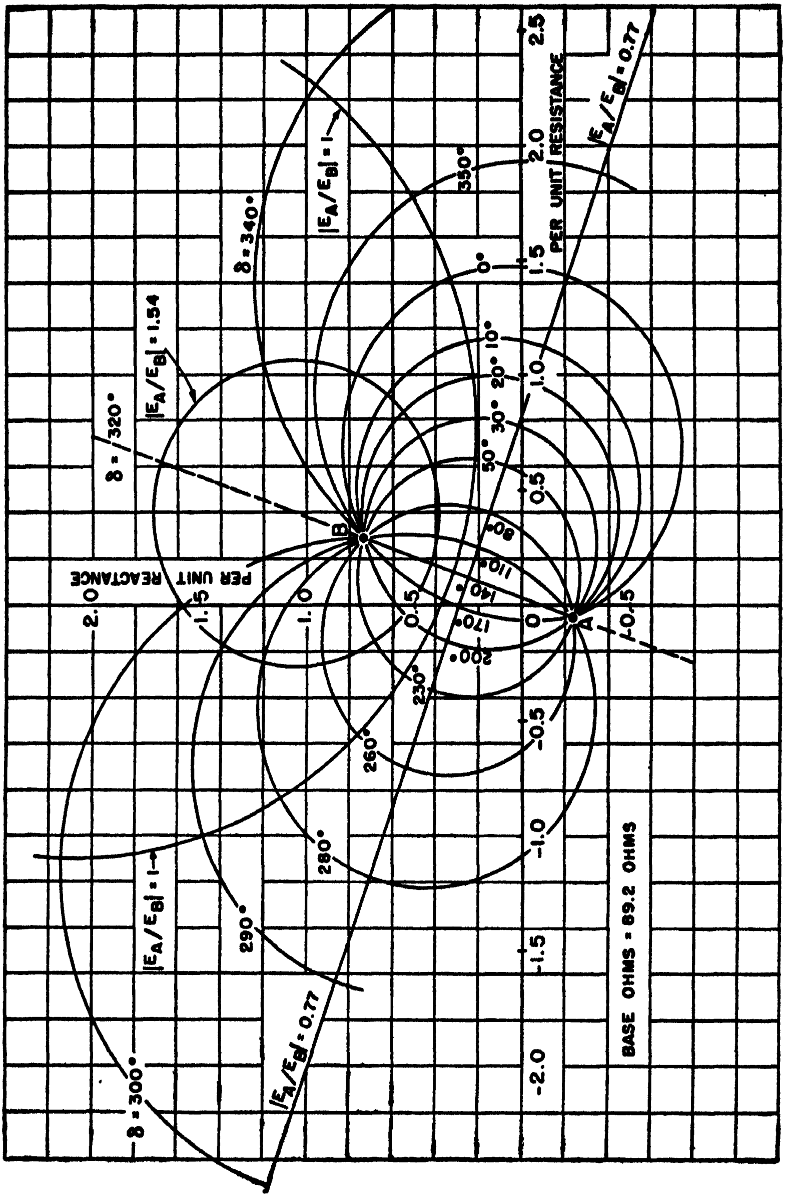


FIG. 8(*b*). Impedance chart for Problem 1. $Z_{ba}$ in per unit of 89.2 ohms. $|E_A/E_B| = 1$.

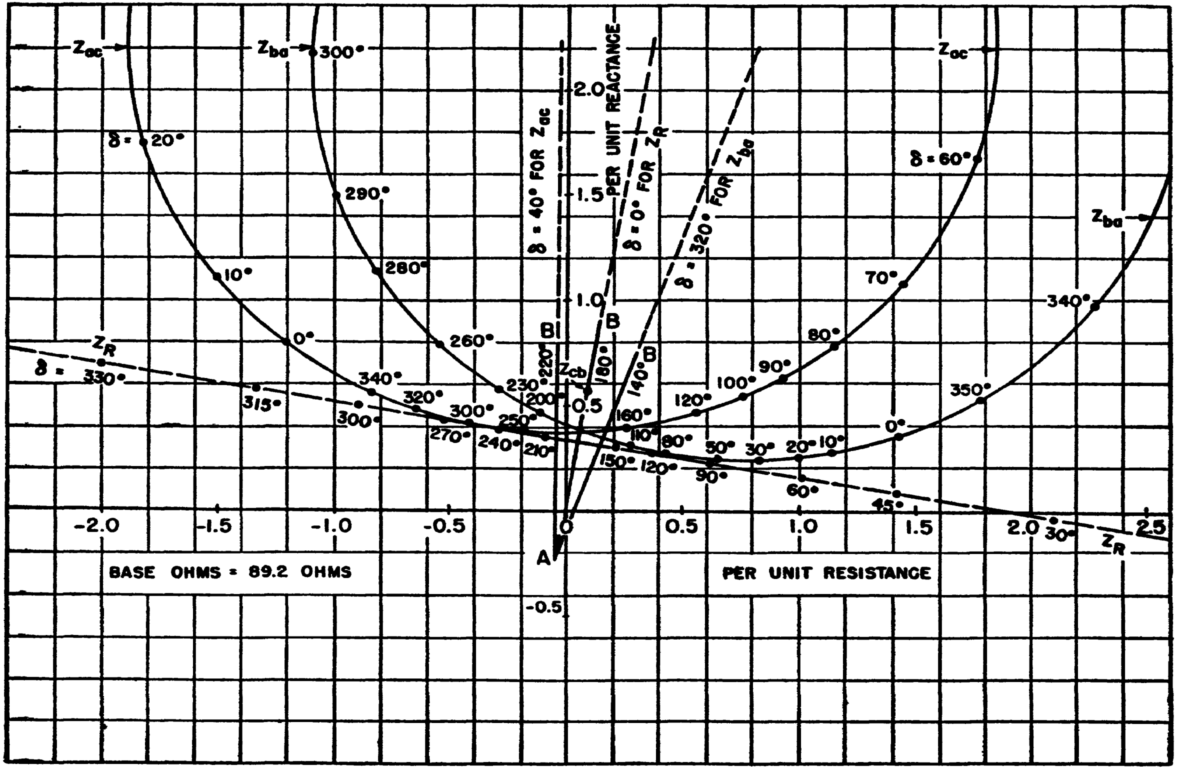


FIG. 8(*c*). Impedance chart for Problem 1. $Z_{cb}$, $Z_{ac}$, $Z_{ba}$ for line-to-line fault, and $Z_R$ with no fault, in per unit of 89.2 ohms. $|E_A/E_B| = 1$.

Let base ohms = 89.2 ohms. In per unit of base ohms, $Z_x = 0.039 + j0.221$; $Z_y = 1.0\underline{/70.48^\circ} = 0.334 + j0.943$; $Z_z = Z_y - Z_x = 0.295 + j0.722$.

The chart for $Z_{ba}$ will be constructed by using the general impedance chart of Fig. 3. In Fig. 8(*b*), the origin $O$ is located and a scale chosen so that $|Z_y| = 1.0$ in per unit of base impedance = the length $AB$ of the general chart. $-Z_x = -0.039 - j0.221$ and $Z_z = 0.295 + j0.722$ are plotted, locating points $A$ and $B$, respectively. The chart of Fig. 8(*b*) is then placed over Fig. 3, with $A$ and $B$ on the two charts coinciding. Various circles and arcs of Fig. 3 are then traced on the chart for $Z_{ba}$. (It is generally unnecessary to trace all circles and arcs completely. Usually the significant portions are obvious, and only those portions need be traced.)

Since in this problem $|K|$ and $\theta_k$ are not 1 and 0°, respectively, the various circles and arcs must be relabeled in values of $|E_A/E_B|$ and $\delta$. For example, the constant $|E_AK/E_B|$ circle in Fig. 3 labeled 1.3, when traced on Fig. 8(*b*), corresponds to $|E_AK/E_B| = 1.3$ or to $|E_A/E_B| = 1.3/|K| = 1.3/1.30 = 1.00$. Similarly, the constant $\phi$ arc in Fig. 3 labeled 90°, when traced on Fig. 8(*b*), corresponds to $\phi = \delta + \theta_k = 90^\circ$ or to $\delta = \phi - \theta_k = 90^\circ - 40^\circ = 50^\circ$.

*Solution* (*b*). Circles for $|E_A/E_B| = 1$ from the charts of Figs. 8(*a*) and 8(*b*) for $Z_{ac}$ and $Z_{ba}$, respectively, are shown on the same chart in Fig. 8(*c*), and the intersections of these circles with various $\delta$ arcs are indicated. The point locating $Z_{cb}$ is also shown. Base impedance in Fig. 8(*c*) is 89.2 ohms.

*Symmetrical System Conditions.* From Table III, case 3, $Z_x = Z_{RR-A} = Z_A = 20\underline{/80^\circ}$ ohms; $Z_z = Z_{RR-B} = Z_B + Z_C = 80\underline{/80^\circ}$ ohms; $Z_y = Z_x + Z_y = 100\underline{/80^\circ}$ ohms; $|K| = 1$; and $\theta_k = 0^\circ$. In per unit of 89.2 ohms — base ohms in Fig. 8(*c*) — $Z_x = 0.2244\underline{/80^\circ} = 0.039 + j0.221$; $Z_y = 1.122\underline{/80^\circ} = 0.195 + j1.105$; $Z_x = Z_y - Z_z = 0.156 + j0.884$.

In Fig. 8(*c*), $-Z_x$ and $Z_z$ are plotted, locating points $A$ and $B$, respectively. The line $AB$ is drawn and its midpoint $M$ determined. The line perpendicular to $AB$ at $M$ is drawn; this line corresponds to $|E_AK/E_B| = 1.0$ and, since $|K| = 1$, it also corresponds to the desired value of $|E_A/E_B| = 1$. The intersections of the $\delta$ arcs with the line for $|E_A/E_B| = 1$ are indicated in Fig. 8(*c*) for $Z_R$ with no fault on the system.

**Problem 2.** (Taken from A.I.E.E. discussion[8] to illustrate advantage of general impedance chart of Fig. 3.) The circuit constants of a transmission line, including transformers, between terminals $A$ and $B$ are: $A = 0.824\underline{/3.25^\circ}$; $B = 282\underline{/77.3^\circ}$ ohms; $C = 1337 \times 10^{-6}\underline{/90.9^\circ}$ mhos; $D = 0.77\underline{/3.6^\circ}$. Determine $Z_x$, $Z_z$, $Z_y$, and $K$, and construct a chart of the system impedances seen by the relays at terminal $A$ during power swings with no fault on the system, using the general impedance chart of Fig. 3.

*Solution.* The voltages at $A$ and $B$ are terminal voltages, not internal generated voltages; case 4 of Table III is applicable, or [47] may be used:

$$Z_{BR} = B = 282\underline{/77.3^\circ} \text{ ohms} \qquad \text{See [30}d\text{].}$$

$$Z_{RR-B} = \frac{B}{D} = 366\underline{/73.7^\circ} = 102 + j351 \text{ ohms} \qquad \text{See [30}b\text{].}$$

From case 4, Table III, or from [47],

$$Z_x = 0$$

$$Z_y = Z_z = Z_{RR-B} = \frac{B}{D} = 102 + j351 \text{ ohms}$$

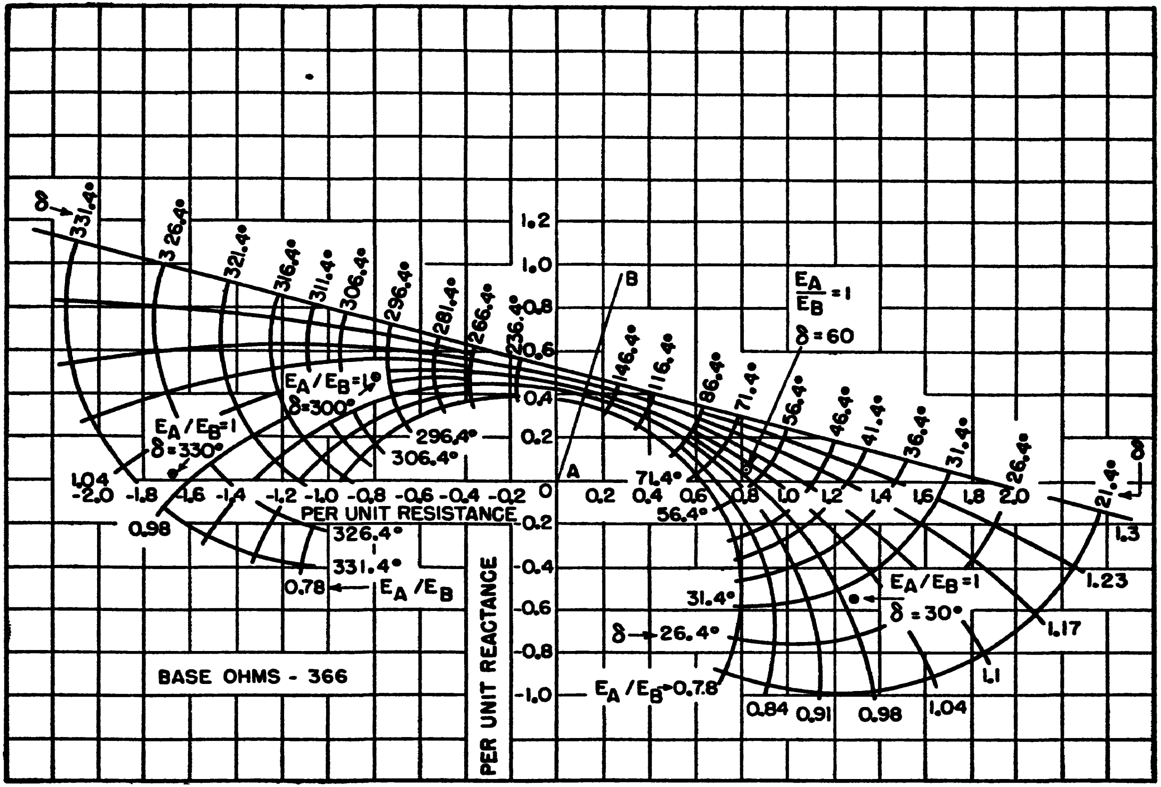


Fig. 9. Impedance chart for Problem 2.

$$K = |K|\underline{/\theta_k} = \frac{Z_{BR}}{Z_{RR-B}} = \frac{282\,\underline{/77.3^\circ}}{366\,\underline{/73.7^\circ}} = D \overset{/}{=} 0.77\,\underline{/3.6^\circ}$$

With $|Z_y| = 1.0$, $Z_z = Z_y = 1\,\underline{/73.7^\circ} = 0.28 + j0.96$; $|K| = 0.77$; $\theta_k = 3.6^\circ$.

A thin piece of graph paper was chosen so that five major divisions corresponded to the distance between points $A$ and $B$ of the general chart of Fig. 3. Resistance and reactance axes were labeled so that the distance between $A$ and $B$ corresponded to 1.0 per unit. Point $A$ ($Z_x = 0$) was located at the origin, and point $B$ ($Z_z = 0.28 + j0.96$) plotted.

The graph paper was placed on top of the general chart of Fig. 3 so that points $A$ and $B$ of the former coincided, respectively, with points $A$ and $B$ of the latter. The sheets were clipped together, and the significant portions of the general chart circles were traced freehand with a pencil.

The constant-voltage-ratio circles were labeled for $E_A/E_B$ by dividing $|E_AK/E_B|$ given on the general chart by $|K|$, and the constant angle arcs were labeled in terms of $\delta$ by subtracting $\theta_k$ from the values of $\phi$ given on the general chart. See Fig. 9. (In this instance, the curves were traced in ink for the purpose of discussion. Ordinarily, if a record is desired, adequate reproductions may be made from the pencil tracing.)

In Fig. 9, four points from Fig. 11 of reference 4 have been plotted, after reduction to per unit values. It will be noted that there is close correspondence between these points obtained by other methods and the results obtained from the general impedance chart.

**Problem 3.** Derive the equations given in Table IV for Fig. 6(*b*).

**Problem 4.** Assuming relays connected for line-to-line voltages and their corresponding delta currents, derive equations for the impedances seen by the three relays at $R$ in Fig. 6(*a*) with a double line-to-ground fault at $F$. Let $Z_0$ = zero-sequence impedance viewed from the fault.

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# APPENDIX A

## TABLE I

### Comparative Wire Gauge Table Giving a Comparison of the Brown and Sharpe, or American (A.W.G.); the Birmingham (B.W.G.); and the British Standard (S.W.G.) Wire Gauges

| Gauge Number | | | Diameter, Inches | | |
|---|---|---|---|---|---|
| A.W.G. | B.W.G. | S.W.G. | A.W.G. | B.W.G. | S.W.G. |
| 40 | | 45 | 0.0031 | | 0.0028 |
| | | 44 | | | 0.0032 |
| 39 | | 43 | 0.0035 | | 0.0036 |
| 38 | 36 | 42 | 0.0040 | 0.004 | 0.0040 |
| 37 | | 41 | 0.0045 | | 0.0044 |
| 36 | 35 | 40 | 0.0050 | 0.005 | 0.0048 |
| 35 | | 39 | 0.0056 | | 0.0052 |
| 34 | | 38 | 0.0063 | | 0.0060 |
| | | 37 | | | 0.0068 |
| 33 | 34 | 36 | 0.0071 | 0.007 | 0.0076 |
| 32 | 33 | 35 | 0.0080 | 0.008 | 0.0084 |
| 31 | 32 | 34 | 0.0089 | 0.009 | 0.0092 |
| 30 | 31 | 33 | 0.0100 | 0.010 | 0.0100 |
| | | 32 | | | 0.0108 |
| 29 | | 31 | 0.0113 | | 0.0116 |
| 28 | 30 | 30 | 0.0126 | 0.012 | 0.0124 |
| | 29 | 29 | | 0.013 | 0.0136 |
| 27 | 28 | 28 | 0.0142 | 0.014 | 0.0148 |
| 26 | 27 | 27 | 0.0159 | 0.016 | 0.0164 |
| 25 | 26 | 26 | 0.0179 | 0.018 | 0.018 |
| 24 | 25 | 25 | 0.0201 | 0.020 | 0.020 |
| 23 | 24 | 24 | 0.0226 | 0.022 | 0.022 |
| 22 | 23 | 23 | 0.0253 | 0.025 | 0.024 |
| 21 | 22 | 22 | 0.0285 | 0.028 | 0.028 |
| 20 | 21 | 21 | 0.0320 | 0.032 | 0.032 |
| 19 | 20 | 20 | 0.0359 | 0.035 | 0.036 |
| 18 | | 19 | 0.0403 | | 0.040 |
| 17 | 19 | | 0.0453 | 0.042 | |
| 16 | 18 | 18 | 0.0508 | 0.049 | 0.048 |
| 15 | 17 | 17 | 0.0571 | 0.058 | 0.056 |
| 14 | 16 | 16 | 0.0641 | 0.065 | 0.064 |
| 13 | 15 | 15 | 0.0720 | 0.072 | 0.072 |
| 12 | 14 | 14 | 0.0808 | 0.083 | 0.080 |
| 11 | 13 | 13 | 0.0907 | 0.095 | 0.092 |
| 10 | 12 | 12 | 0.1019 | 0.109 | 0.104 |
| 9 | 11 | 11 | 0.1144 | 0.120 | 0.116 |
| 8 | 10 | 10 | 0.1285 | 0.134 | 0.128 |
| 7 | 9 | 9 | 0.1443 | 0.148 | 0.144 |
| 6 | 8 | 8 | 0.1620 | 0.165 | 0.160 |
| 5 | 7 | 7 | 0.1819 | 0.180 | 0.176 |
| 4 | 6 | 6 | 0.2043 | 0.203 | 0.192 |
| | 5 | 5 | | 0.220 | 0.212 |
| 3 | 4 | 4 | 0.2294 | 0.238 | 0.232 |
| 2 | 3 | 3 | 0.2576 | 0.259 | 0.252 |
| 1 | 2 | 2 | 0.2893 | 0.284 | 0.276 |
| | 1 | 1 | | 0.300 | 0.300 |
| 1/0 | | 1/0 | 0.3249 | | 0.324 |
| | 1/0 | 2/0 | | 0.340 | 0.348 |
| 2/0 | 2/0 | 3/0 | 0.3648 | 0.380 | 0.372 |
| 3/0 | | 4/0 | 0.4096 | | 0.400 |
| | 3/0 | 5/0 | | 0.425 | 0.432 |
| 4/0 | 4/0 | 6/0 | 0.4600 | 0.454 | 0.464 |

## TABLE II

### SOLID CONDUCTORS OF STANDARD ANNEALED COPPER
### (U.S. Bureau of Standards)

| Gauge No., A.W.G.* | Diameter, Mils | Cross Section | | Ohms per 1000 Ft at 25°C†(77°F) | Pounds per 1000 Ft |
|---|---|---|---|---|---|
| | | Circular Mils | Square Inches | | |
| 40 | 3.1 | 9.9 | 0.0000078 | 1,070 | 0.0299 |
| 39 | 3.5 | 12.5 | 0.0000098 | 848 | 0.0377 |
| 38 | 4.0 | 15.7 | 0.0000123 | 673 | 0.0476 |
| 37 | 4.5 | 19.8 | 0.0000156 | 533 | 0.0600 |
| 36 | 5.0 | 25.0 | 0.0000196 | 423 | 0.0757 |
| 35 | 5.6 | 31.5 | 0.0000248 | 336 | 0.0954 |
| 34 | 6.3 | 39.8 | 0.0000312 | 266 | 0.120 |
| 33 | 7.1 | 50.1 | 0.0000394 | 211 | 0.152 |
| 32 | 8.0 | 63.2 | 0.0000496 | 167 | 0.191 |
| 31 | 8.9 | 79.7 | 0.0000626 | 133 | 0.241 |
| 30 | 10.0 | 101.0 | 0.0000789 | 105 | 0.304 |
| 29 | 11.3 | 127 | 0.0000995 | 83.5 | 0.384 |
| 28 | 12.6 | 160 | 0.000126 | 66.2 | 0.484 |
| 27 | 14.2 | 202 | 0.000158 | 52.5 | 0.610 |
| 26 | 15.9 | 254 | 0.000200 | 41.6 | 0.769 |
| 25 | 17.9 | 320 | 0.000252 | 33.0 | 0.970 |
| 24 | 20.1 | 404 | 0.000317 | 26.2 | 1.22 |
| 23 | 22.6 | 509 | 0.000400 | 20.8 | 1.54 |
| 22 | 25.3 | 642 | 0.000505 | 16.5 | 1.94 |
| 21 | 28.5 | 810 | 0.000636 | 13.1 | 2.45 |
| 20 | 32 | 1,020 | 0.000802 | 10.4 | 3.09 |
| 19 | 36 | 1,290 | 0.00101 | 8.21 | 3.90 |
| 18 | 40 | 1,620 | 0.00128 | 6.51 | 4.92 |
| 17 | 45 | 2,050 | 0.00161 | 5.16 | 6.20 |
| 16 | 51 | 2,580 | 0.00203 | 4.09 | 7.82 |
| 15 | 57 | 3,260 | 0.00256 | 3.25 | 9.86 |
| 14 | 64 | 4,110 | 0.00323 | 2.58 | 12.4 |
| 13 | 72 | 5,180 | 0.00407 | 2.04 | 15.7 |
| 12 | 81 | 6,530 | 0.00513 | 1.62 | 19.8 |
| 11 | 91 | 8,230 | 0.00647 | 1.28 | 24.9 |
| 10 | 102 | 10,400 | 0.00815 | 1.02 | 31.4 |
| 9 | 114 | 13,100 | 0.0103 | 0.808 | 39.6 |
| 8 | 128 | 16,500 | 0.0130 | 0.641 | 50.0 |
| 7 | 144 | 20,800 | 0.0164 | 0.508 | 63.0 |
| 6 | 162 | 26,300 | 0.0206 | 0.403 | 79.5 |
| 5 | 182 | 33,100 | 0.0260 | 0.320 | 100 |
| 4 | 204 | 41,700 | 0.0328 | 0.253 | 126 |
| 3 | 229 | 52,600 | 0.0413 | 0.201 | 159 |
| 2 | 258 | 66,400 | 0.0521 | 0.159 | 201 |
| 1 | 289 | 83,700 | 0.0657 | 0.126 | 253 |
| 0 | 325 | 106,000 | 0.0829 | 0.100 | 319 |
| 00 | 365 | 133,000 | 0.105 | 0.0795 | 403 |
| 000 | 410 | 168,000 | 0.132 | 0.0630 | 508 |
| 0000 | 460 | 212,000 | 0.166 | 0.0500 | 641 |

* American Wire Gauge.

† D-c resistance calculated at 25°C from [17], Chapter II, with dimensions given at 20°C. The d-c resistance $R_t$ of a given conductor at temperature $t$°C is

$$R_t = R_{25}[1 + 0.00385(t - 25)]$$

where $R_{25}$ is the resistance at 25°C read from the table.

## TABLE III

### Bare Concentric Lay Cables of Standard Annealed Copper
### (U.S. Bureau of Standards)*

| Size, A.W.G. or 1000 Cir Mils | Ohms per 1000 Ft at 25°C† (77°F) | Pounds per 1000 Ft | Standard Strands | | Flexible Strands | | Outside Diameter, Mils | Resistance-Skin Effect Ratio at 65°C | |
|---|---|---|---|---|---|---|---|---|---|
| | | | No. of Wires | Diameter of Wires, Mils | No. of Wires | Diameter of Wires, Mils | | 25 Cycles | 60 Cycles |
| 16 | 4.176 | 8 | 7 | 19.2 | 19 | 11.7 | 58 | | |
| 14 | 2.626 | 12.7 | 7 | 24.2 | 19 | 14.7 | 73 | | |
| 12 | 1.652 | 20 | 7 | 30.5 | 19 | 18.5 | 92 | | |
| 10 | 1.039 | 32 | 7 | 38.5 | 19 | 23.4 | 117 | | |
| 8 | 0.653 | 51 | 7 | 48.6 | 19 | 29.5 | 147 | | |
| 7 | 0.518 | 64.3 | 7 | 54.5 | 19 | 33.1 | 166 | | |
| 6 | 0.411 | 81 | 7 | 61.2 | 19 | 37.2 | 186 | | |
| 5 | 0.326 | 102 | 7 | 68.8 | 19 | 41.7 | 209 | | |
| 4 | 0.258 | 129 | 7 | 77.2 | 19 | 46.9 | 234 | | |
| 3 | 0.205 | 163 | 7 | 86.7 | 19 | 52.6 | 263 | | |
| 2 | 0.163 | 205 | 7 | 97.4 | 19 | 59.1 | 296 | | |
| 1 | 0.129 | 258 | 19 | 66.4 | 37 | 47.6 | 333 | | |
| 0 | 0.102 | 326 | 19 | 74.5 | 37 | 53.4 | 374 | | |
| 00 | 0.0811 | 411 | 19 | 83.7 | 37 | 60.0 | 420 | | |
| 000 | 0.0643 | 518 | 19 | 94.0 | 37 | 67.3 | 471 | | |
| 0000 | 0.0510 | 653 | 19 | 105.5 | 37 | 75.6 | 533 | | |
| 250 | 0.0432 | 772 | 37 | 82.2 | 61 | 64.0 | 576 | | 1.005 |
| 300 | 0.0360 | 926 | 37 | 90.0 | 61 | 70.1 | 631 | | 1.006 |
| 350 | 0.0308 | 1080 | 37 | 97.3 | 61 | 75.7 | 682 | | 1.009 |
| 400 | 0.0270 | 1240 | 37 | 104.0 | 61 | 81.0 | 729 | | 1.011 |
| 450 | 0.0240 | 1390 | 37 | 110.3 | 61 | 85.9 | 773 | | 1.014 |
| 500 | 0.0216 | 1540 | 37 | 116.2 | 61 | 90.5 | 815 | | 1.018 |
| 550 | 0.0196 | 1700 | 61 | 95.0 | 91 | 77.7 | 855 | | 1.021 |
| 600 | 0.0180 | 1850 | 61 | 99.2 | 91 | 81.2 | 893 | 1.005 | 1.025 |
| 650 | 0.0166 | 2010 | 61 | 103.2 | 91 | 84.5 | 930 | 1.005 | 1.029 |
| 700 | 0.0154 | 2160 | 61 | 107.1 | 91 | 87.7 | 965 | 1.006 | 1.034 |
| 750 | 0.0144 | 2320 | 61 | 110.9 | 91 | 90.8 | 999 | 1.007 | 1.039 |
| 800 | 0.0135 | 2470 | 61 | 114.5 | 91 | 93.8 | 1031 | 1.008 | 1.044 |
| 850 | 0.0127 | 2620 | 61 | 118.0 | 91 | 96.6 | 1063 | 1.009 | 1.049 |
| 900 | 0.0120 | 2780 | 61 | 121.5 | 91 | 99.4 | 1094 | 1.010 | 1.055 |
| 950 | 0.0114 | 2930 | 61 | 124.8 | 91 | 102.2 | 1124 | 1.011 | 1.061 |
| 1000 | 0.0108 | 3090 | 61 | 128.0 | 91 | 104.8 | 1153 | 1.012 | 1.067 |
| 1250 | 0.00863 | 3860 | 91 | 117.2 | 127 | 99.2 | 1289 | 1.019 | 1.102 |
| 1500 | 0.00719 | 4630 | 91 | 128.4 | 127 | 108.7 | 1413 | 1.027 | 1.142 |
| 1750 | 0.00619 | 5410 | 127 | 117.4 | 169 | 101.8 | 1527 | 1.037 | 1.185 |
| 2000 | 0.00539 | 6180 | 127 | 125.5 | 169 | 108.8 | 1632 | 1.048 | 1.233 |

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* The values given for ohms per 1000 feet and pounds per 1000 ft are 2% greater than for a solid rod of cross section equal to the total cross section of the wires of the cable.

† See footnote to Table II for calculation of resistance at other temperatures.

## TABLE IV

### ANNULAR CONDUCTOR CABLE*

### (Stranded Annular Conductors of Standard Annealed Copper)

| Nominal Size, Cir Mils | Approx. Rope Size, Inches | Diameter of Wires, Inches | No. of Strands in Each Layer | | | Total | Actual Circular Mils Area | Max. O.D., Inches | Approx. Pounds per 1000 Ft | | Approx. D-C Resistance, Ohms per 1000 Ft† | | Resistance-Skin Effect Ratio at 65°C | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st | 2nd | 3rd | | | | Copper | Copper and Core | 25°C | 65°C | 25 Cycles | 60 Cycles |
| 750,000 | 0.375 | 0 1172 | 12 | 18 | 24 | 54 | 741,735 | 1 108 | 2,312 | 2,362 | 0 0147 | 0 0169 | 1 004 | 1 021 |
| 800,000 | 0 468 | 0 1110 | 16 | 21 | 28 | 65 | 800,865 | 1 164 | 2,497 | 2,575 | 0 0136 | 0 0157 | 1 004 | 1 020 |
| 900,000 | 0 500 | 0 1172 | 16 | 22 | 28 | 66 | 906,565 | 1 234 | 2,826 | 2,905 | 0 0120 | 0 0139 | 1 005 | 1 025 |
| 1,000,000 | 0 563 | 0 1255 | 16 | 21 | 28 | 65 | 1,023,766 | 1 346 | 3,192 | 3,304 | 0 0106 | 0 0123 | 1 006 | 1 031 |
| 1,250,000 | 0.750 | 0 1255 | 21 | 26 | 33 | 80 | 1,260,020 | 1 533 | 3,928 | 4,127 | 0 00864 | 0 0100 | 1 007 | 1.034 |
| 1,500,000 | 1 000 | 0 1255 | 26 | 32 | 38 | 96 | 1,512,024 | 1 783 | 4,714 | 5,068 | 0 00720 | 0 00831 | 1 007 | 1.037 |
| 1,750,000 | 1 125 | 0 1280 | 30 | 35 | 42 | 107 | 1,753,088 | 1 923 | 5,466 | 5,914 | 0 00621 | 0 00717 | 1 008 | 1.043 |
| 2,000,000 | 1 3125 | 0 1284 | 34 | 40 | 46 | 120 | 1,978,387 | 2 114 | 6,168 | 6,778 | 0 00550 | 0 00635 | 1 009 | 1 045 |
| 2,500,000 | 1 500 | 0 1440 | 34 | 40 | 46 | 120 | 2,488,320 | 2 394 | 7,758 | 8,555 | 0 00438 | 0 00505 | 1 012 | 1 066 |
| 3,000,000 | 1 625 | 0 1620 | 33 | 38 | 45 | 116 | 3,044,304 | 2 627 | 9,492 | 10,427 | 0 00358 | 0 00413 | 1 019 | 1.105 |
| 3,500,000 | 2 000 | 0 1620 | 40 | 45 | 52 | 137 | 3,595,428 | 3 007 | 11,319 | 12,735 | 0 00306 | 0 00353 | 1 020 | 1.110 |
| 4,000,000 | 2 250 | 0 1620 | 45 | 51 | 57 | 153 | 4,015,332 | 3 262 | 12,641 | 14,433 | 0 00274 | 0 00316 | 1 021 | 1.116 |
| 4,500,000 | 2 500 | 0 1620 | 50 | 56 | 62 | 168 | 4,408,992 | 3 517 | 14,013 | 16,226 | 0 00252 | 0 00291 | 1 022 | 1.118 |
| 5,000,000 | 2 875 | 0 1620 | 57 | 63 | 69 | 189 | 4,960,116 | 3 897 | 15,765 | 18,691 | 0 00224 | 0 00258 | 1 023 | 1.121 |

The following plus tolerances are included in maximum outside diameters:

| Nominal Size in Cir Mils | Tolerance in Inches |
|---|---|
| 750,000 to 3,000,000 | 0 030 |
| 3,001,000 to 3,500,000 | 0 035 |
| 3,501,000 to 4,000,000 | 0 040 |
| 4,001,000 to 4,500,000 | 0 045 |
| 4,501,000 to 5,000,000 | 0 050 |

The following increases for stranding are included in the metallic weight and d-c resistance data·

| Nominal Size in Cir Mils | Increase Per Cent |
|---|---|
| 750,000 to 3,000,000 | 3 |
| 3,001,000 to 4,000,000 | 4 |
| 4,001,000 to 5,000,000 | 5 |

The weight of the core used is the saturated weight. For dry core deduct from the copper and core weight 20% of the difference between the two weights given in the table.

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* See reference 4, Chapter III.

† See footnote, Table II

## TABLE V

### RECOMMENDED RUBBER INSULATION THICKNESS*

| Rated† Circuit Voltage, Volts | Size‡ of Conductor A.W.G. or M.C.M. | Nominal Thickness of Insulation in 64ths Inch | |
|---|---|---|---|
| | | Grounded Neutral§ | Ungrounded Neutral |
| 600 | 14–9 | 3 | 3 |
| | 8–2 | 4 | 4 |
| | 1–4/0 | 5 | 5 |
| | 225–500 | 6 | 6 |
| | 525–1000 | 7 | 7 |
| | Over 1000 | 8 | 8 |
| 2,400 | 10–8 | 7 | 7 |
| | 7–4/0 | 8 | 8 |
| | 225–1000 | 9 | 9 |
| | Over 1000 | 10 | 10 |
| 4,160–4,800 | 8–4/0 | 10 | 10 |
| | 225–1000 | 11 | 11 |
| | Over 1000 | 12 | 12 |
| 7,200 | 8–1000 | 12 | 16 |
| | Over 1000 | 13 | 17 |
| 8,320 | 6–1000 | 13 | 17 |
| | Over 1000 | 14 | 18 |
| 12,000 | 6–1000 | 16 | 22 |
| | Over 1000 | 17 | 23 |
| 15,000 | 6–1000 | 19 | 27 |
| | Over 1000 | 20 | 28 |
| 23,000 | 2–1000 | 27 | |
| | Over 1000 | 28 | |
| 27,000 | 1–1000 | 31 | |
| | Over 1000 | 32 | |

For non-leaded submarine cables $\frac{1}{32}$ in. shall be added to the specified respective walls of insulation in Table V for all sizes and voltages.

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* See reference 3, Chapter III.

† Rated voltage is phase-to-phase voltage. The voltages listed are the " Preferred Voltage Ratings for Power Cables " given in the EEI-NEMA report, " Preferred Voltage Ratings for A-C Systems and Equipment," May 1949.

‡ The first conductor size given for each circuit-voltage class is the minimum size recommended for that voltage class.

§ A circuit is considered to have a grounded neutral if the neutral is permanently connected to earth and if facilities are provided to insure prompt isolation of a faulty element of the circuit.

## TABLE VI

### RECOMMENDED THICKNESS OF VARNISHED CAMBRIC INSULATION*

(Single-Conductor Cable and Multiple-Conductor Shielded Cable)

| Rated† Voltage, Volts, Phase to Phase | Size,‡ A.W.G. or 1000 Cir Mils | Wall of Varnished Cambric in 64ths Inch (and Mils) | |
|---|---|---|---|
| | | Grounded Neutral§ | Ungrounded Neutral |
| 600 | 14–8 | 3 (47) | 3 (47) |
| | 7–2 | 4 (63) | 4 (63) |
| | 1–4/0 | 5 (78) | 5 (78) |
| | 213–500 | 6 (94) | 6 (94) |
| | 501–1000 | 7 (109) | 7 (109) |
| | 1001 and larger | 8 (125) | 8 (125) |
| 2,400 | 10–2 | 6 (94) | 6 (94) |
| | 1–4/0 | 6 (94) | 6 (94) |
| | 213–500 | 7 (109) | 7 (109) |
| | 501–1000 | 7 (109) | 7 (109) |
| | 1001 and larger | 8 (125) | 8 (125) |
| 4,160–4,800 | 8–4/0 | 8 (125) | 9 (141) |
| | 213–1000 | 9 (141) | 10 (156) |
| | 1001 and larger | 9 (141) | 10 (156) |
| 7,200 | 6 and larger | 11 (172) | 12 (188) |
| 8,320 | 6 and larger | 12 (188) | 13 (203) |
| 12,000 | 6 and larger | 14 (219) | 16 (250) |
| 15,000 | 6 and larger | 16 (250) | 21 (328) |
| 23,000 | 2 and larger | 24 (375) | |
| 27,000 | 2 and larger | 28 (438) | |

* See reference 4, Chapter III.

†, ‡, § See footnotes to Table V.

## TABLE VII

### Recommended Thickness of Varnished Cambric Insulation*

(Multiple-Conductor Belted Cables)

| Rated† Voltage, Volts, Phase to Phase | Size,‡ A.W.G. or 1000 Cir Mils | Wall of Varnished Cambric in 64ths Inch (and Mils) | | | |
|---|---|---|---|---|---|
| | | Grounded Neutral§ | | Ungrounded Neutral | |
| | | On Conductors | On Belt | On Conductors | On Belt |
| 600 | 14–8 | 3 (47) | 0 (0) | 3 (47) | 0 (0) |
| | 7–2 | 4 (63) | 0 (0) | 4 (63) | 0 (0) |
| | 1–4/0 | 5 (78) | 0 (0) | 5 (78) | 0 (0) |
| | 213–500 | 6 (94) | 0 (0) | 6 (94) | 0 (0) |
| | 501–1000 | 6 (94) | 2 (31) | 6 (94) | 2 (31) |
| | 1001 and larger | 7 (109) | 2 (31) | 7 (109) | 2 (31) |
| 2,400 | 10–2 | 5 (78) | 2 (31) | 5 (78) | 2 (31) |
| | 1–4/0 | 6 (94) | 2 (31) | 6 (94) | 2 (31) |
| | 213–500 | 6 (94) | 2 (31) | 6 (94) | 2 (31) |
| | 501–1000 | 6 (94) | 3 (47) | 6 (94) | 3 (47) |
| | 1001 and larger | 7 (109) | 3 (47) | 7 (109) | 3 (47) |
| 4,160–4,800 | 8–4/0 | 6 (94) | 4 (63) | 6 (94) | 4 (63) |
| | 213–1000 | 7 (109) | 4 (63) | 7 (109) | 4 (63) |
| | 1001 and larger | 7 (109) | 5 (78) | 7 (109) | 5 (78) |
| 7,200 | 6 and larger | 7 (109) | 6 (94) | 7 (109) | 7 (109) |
| 8,320 | 6 and larger | 8 (125) | 6 (94) | 8 (125) | 8 (125) |
| 12,000 | 6 and larger | 10 (156) | 7 (109) | 10 (156) | 10 (156) |
| 15,000 | 6 and larger | 13 (203) | 7 (109) | 13 (203) | 13 (203) |

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* See reference 4, Chapter III.

†, ‡, § See footnotes to Table V.

## TABLE VIII

### RECOMMENDED THICKNESS OF INSULATION* OF SOLID-PAPER SINGLE-CONDUCTOR CABLE, SHIELDED AND NON-SHIELDED TYPES

(Grounded Neutral†)

| Rated‡ Voltage, Volts, Phase to Phase | Thickness of Insulation, Mils | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Size§ of Conductors, A.W.G. or M.C.M. | | | | | | | | | | | | | | |
| | 8–7 | 6–5 | 4 | 3 | 2 | 1 | 1/0 | 2/0 | 3/0 | 4/0 | 250 | 300 | 350–1000 | 1001–2500 | 2501–4000 |
| 600 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 75 | 90 |
| 2,400 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 90 | 105 |
| 4160-4800 | 120 | 120 | 115 | 115 | 110 | 110 | 105 | 100 | 100 | 95 | 90 | 90 | 90 | 100 | 110 |
| 7,200 | | 150 | 150 | 145 | 140 | 140 | 135 | 130 | 125 | 120 | 120 | 115 | 115 | 115 | 115 |
| 8,320 | | 160 | 160 | 155 | 150 | 150 | 145 | 140 | 135 | 130 | 125 | 125 | 125 | 125 | 125 |
| 12,000 | | 195 | 190 | 185 | 185 | 180 | 170 | 165 | 160 | 155 | 150 | 150 | 150 | 150 | 150 |
| 15,000 | | | 220 | 220 | 215 | 210 | 200 | 195 | 185 | 180 | 175 | 175 | 175 | 175 | 175 |
| 23,000 | | | | | 295 | 285 | 275 | 265 | 260 | 250 | 245 | 240 | 240 | 240 | 240 |
| 27,000 | | | | | | 325 | 315 | 305 | 295 | 280 | 275 | 275 | 275 | 275 | 275 |
| 35,000 | | | | | | | 395 | 385 | 370 | 355 | 350 | 345 | 345 | 345 | 345 |
| 46,000 | | | | | | | | | 475 | 455 | 450 | 445 | 445 | 445 | 445 |
| 69,000 | | | | | | | | | | | | | 650 | 650 | 650 |

* See reference 5, Chapter III.

†, ‡, § See footnotes to Table V.

## TABLE IX

### RECOMMENDED THICKNESS OF INSULATION* OF SOLID-PAPER MULTIPLE-CONDUCTOR CABLE, SHIELDED TYPE

(Grounded Neutral†)

| Rated‡ Voltage, Volts, Phase to Phase | Thickness of Insulation for Round Conductors, Mils | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Size§ of Conductors, A.W.G. or M.C.M. | | | | | | | | |
| | 6 | 5 | 4 | 3 | 2 | 1 | 1/0 | 2/0 | 3/0–1000 |
| 7,200 and below | 145 | 140 | 135 | 135 | 130 | 120 | 120 | 115 | 115 |
| 8,320 | 155 | 150 | 145 | 140 | 135 | 130 | 130 | 125 | 125 |
| 12,000 | 185 | 180 | 175 | 170 | 165 | 155 | 155 | 150 | 150 |
| 15,000 | | | 205 | 200 | 190 | 185 | 180 | 175 | 175 |
| 23,000 | | | | | 265 | 255 | 250 | 240 | 240 |
| 27,000 | | | | | | 290 | 280 | 275 | 275 |
| 35,000 | | | | | | | 355 | 345 | 345 |
| 46,000 | | | | | | | | | 445 |

* See reference 5, Chapter III.

†‡ See footnotes to Table V.

§ The first conductor size given for each voltage class is the minimum size recommended for that voltage class for round conductors. When sector conductors are used, the same data apply except that the minimum sizes are given in the table below:

| Rated Kv | Minimum Size, A.W.G. or M.C.M. |
|---|---|
| 8–27 | 2/0 |
| 35 | 250 |
| 46 | 350 |

# TABLE X

## Recommended Thickness of Insulation* of Solid-Paper -- Multiple-Conductor Cable, Belted Type

(Grounded Neutral†)

(Insulation Thickness Values are for Conductor and Belt, Respectively)

| Rated‡ Voltage, Volts, Phase to Phase | Thickness of Insulation for Round Conductors, Mils | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Size§ of Conductors, A.W.G. or M.C.M. | | | | | | | | | | |
| | 8-7 | 6 | 5 | 4 | 3 | 2 | 1 | 1/0 | 2/0-500 | 501-750 | Over 750 |
| 600 | 55-35 | 55-35 | 55-35 | 55-35 | 55-35 | 55-35 | 55-35 | 55-35 | 55-35 | 60-40 | 75-45 |
| 2,400 | 70-40 | 70-40 | 70-40 | 70-40 | 70-40 | 70-40 | 70-40 | 70-40 | 70-40 | 75-40 | 80-45 |
| 4,160-4,800 | 110-60 | 110-55 | 105-55 | 100-55 | 95-50 | 95-50 | 90-45 | 90-45 | 85-45 | 85-45 | 85-45 |
| 7.200 | | 130-65 | 130-65 | 125-65 | 120-60 | 115-60 | 110-55 | 110-55 | 105-55 | 105-55 | 105-55 |
| 8,320 | | 140-70 | 135-70 | 135-65 | 130-65 | 125-65 | 120-60 | 115-60 | 115-55 | 115-55 | 115-55 |
| 12,000 | | | | 160-80 | 155-75 | 145-75 | 140-70 | 140-70 | 135-65 | 135-65 | 135-65 |
| 15,000 | | | | 185-90 | 175-85 | 170-85 | 165-80 | 160-75 | 155-75 | 155-75 | 155-75 |

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* See reference 5, Chapter III.

†‡ See footnotes to Table V.

§ The first conductor size given for each voltage class is the minimum size recommended for that voltage class for round conductors. When sector conductors are used, the same data apply except that the minimum sizes are given in the table below:

| Rated Kv | Minimum Size, A.W.G. or M.C.M. |
|---|---|
| 1-5 | 1/0 |
| 6-9 | 2/0 |
| 10-12 | 3/0 |
| 13-15 | 350 |

## TABLE XI

### THICKNESS OF INSULATION* FOR OIL-FILLED CABLE
### (Grounded Neutral†)

| Rated‡ Voltage, Kv, Phase to Phase | Single-Conductor Cable | | Three-Conductor Cable | |
|---|---|---|---|---|
| | Size§ of Conductors, A.W.G. or M.C.M. | Thickness of Insulation,‖ Mils | Size§ of Compact Round Conductors, A.W.G. or M.C.M. | Thickness of Insulation, Mils |
| 15 | 1/0–1000 | 110 | 1–750 | 110 |
| | 1001–2500 | 125 | 751–1000 | 125 |
| 23 | 1/0–2500 | 145 | 1–750 | 145 |
| | | | 751–1000 | 160 |
| 27 | 1/0–2500 | 160 | 1–750 | 160 |
| | | | 751–1000 | 175 |
| 35 | 1/0–2500 | 190 | 1/0–750 | 190 |
| | | | 751–1000 | 200 |
| 46 | 2/0–2500 | 225 | 1/0–1000 | 225 |
| 69 | 2/0–2500 | 315 | 2/0–1000 | 315 |
| 115 | 4/0–2500 | 480 | | |
| 138 | 4/0–2500 | 560 | | |
| 161 | 250–2500 | 650 | | |
| 230 | 750–2500 | 925 | | |

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* See reference 6, Chapter III.

†‡ See footnotes to Table V.

§ The first conductor size given for each voltage class is the minimum size recommended for the voltage class for round conductors. When sector conductors are used, the same data apply except that the minimum size must be somewhat larger.

‖ The thicknesses given here are applicable to hollow-core conductors with either of the two standard internal diameters, i.e., 0.50 in. or 0.69 in.

## TABLE XII

### MAXIMUM ALLOWABLE COPPER TEMPERATURE OF INSULATED CABLES

RUBBER COMPOUNDS*

| Type of Compound | Voltage Rating | Maximum Copper Temperature, °C |
|---|---|---|
| Standard compounds | | |
| NEMA Code and Intermediate Grades | Up to 5 kv | 50 |
| NEMA and ASTM Standard 30% and Performance Grades | Up to 8 kv | 60 |
| 34 to 60% compounds | Up to 8 kv | 60 |
| Special compounds recommended by manufacturer: | | |
| All special compounds other than heat-resisting | Any voltage | 60 |
| Heat-resisting compounds | Up to 8 kv | 75 |
| Heat-resisting compounds | Above 8 kv | 70 |

VARNISHED CAMBRIC†

| Rated Voltage, Volts, Phase to Phase | Maximum Safe Copper Temperature, °C | |
|---|---|---|
| | Single-Conductor and Multiple-Conductor Shielded Cables‡ | Multiple-Conductor Belted Cables |
| 0– 5,000 | 85 | 85 |
| 5,001– 6,000 | 85 | 83 |
| 6,001– 7,000 | 84 | 82 |
| 7,001– 7,500 | 84 | 81 |
| 7,501– 8,000 | 83 | 80 |
| 8,001– 9,000 | 82 | 78 |
| 9,001–10,000 | 81 | 76 |
| 10,001–11,000 | 80 | 75 |
| 11,001–12,000 | 80 | 73 |
| 12,001–13,000 | 79 | 71 |
| 13,001–14,000 | 78 | 70 |
| 14,001–15,000 | 77 | 70 |
| 15,001–16,000 | 76 | 70 |
| 16,001–17,000 | 75 | 70 |
| 17,001–18,000 | 74 | |
| 18,001–19,000 | 74 | |
| 19,001–20,000 | 73 | |
| 20,001–21,000 | 72 | |
| 21,001–22,000 | 71 | |
| 22,001–23,000 | 70 | |
| 23,001–24,000 | 70 | |
| 24,001–25,000 | 70 | |
| 25,001–26,000 | 70 | |
| 26,001–27,000 | 70 | |
| 27,001–28,000 | 70 | |

See footnotes on following page.

## TABLE XII (*Continued*)

### IMPREGNATED PAPER, SOLID TYPE†

| Rated Voltage, Kilovolts, Phase to Phase | Maximum Safe Copper Temperature, °C | | |
|---|---|---|---|
| | Single-Conductor Cables | Multiple-Conductor Cables | |
| | | Shielded | Belted |
| 0–1.0 | 85 | | 85 |
| 3.0 | 85 | | 85 |
| 5.0 | 85 | | 85 |
| 7.0 | 85 | | 83 |
| 9.0 | 85 | 85 | 81 |
| 12.0 | 83 | 83 | 78 |
| 15 | 81 | 81 | 75 |
| 27.0 | 74 | 74 | |
| 35 | 70 | 70 | |
| 46 | 63 | 63 | |
| 69 | 60 | | |

The temperatures for intermediate ratings shall be determined by interpolation. Cable permanently operated at less than its rated voltage may be operated at the temperature corresponding to the operating voltage except that such increase in temperature shall not exceed 10° C.

### IMPREGNATED PAPER, OIL-FILLED TYPE†

| Rated Voltage in Kilovolts, Phase to Phase | Maximum Safe Copper Temperature, °C |
|---|---|
| 15 to 25 | Same as single-conductor, solid-type cable specifications |
| 26 to 69 | 75 |
| 70 to 230 | 70 |

These temperatures apply to continuous operation. A tolerance of 10°C shall be allowed over and above these guaranteed temperatures, provided that the total time the temperatures are exceeded is not greater than 100 hr in any one year. These are limits for the paper in the insulation. Local conditions may reduce the limits slightly on the basis of the ability of the sheath to withstand repeated bending due to expansion and contraction.

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[*] "Current Carrying Capacity of Rubber Insulated Wire," IPCEA Standard P-19-102, 1941.

[†] See references 4, 5, and 6, Chapter III.

[‡] With individually shielded conductors, type H.

# APPENDIX B

## DEVELOPMENT OF EQUATIONS — ROTATING MACHINES

In analyzing the performance of rotating machines, well-known relations can often be stated without proof, or with references to previously published papers or books. This has the advantage of permitting a more concise treatment but, in cases where references are scattered and not easily obtained, statements not accompanied by proof are unsatisfactory to the serious student who has forgotten or was never familiar with the proof. For the benefit of such students, and to avoid the introduction of reading matter unnecessary for the highly technical, certain statements are given without proof in the chapters on induction and synchronous machines with references to this appendix.

**Armature Current Phenomena.** In a three-phase synchronous or induction machine with positive-sequence currents of fundamental frequency only in the armature, the instantaneous currents $i_a$, $i_b$, and $i_c$ in phases $a$, $b$, and $c$, respectively, are

$$i_a = I_m \cos t; \quad i_b = I_m \cos\left(t - \frac{2\pi}{3}\right); \quad i_c = I_m \cos\left(t + \frac{2\pi}{3}\right)$$

where phase order is $abc$, $I_m$ is the amplitude of the positive-sequence current and $t$ is time in electrical radians reckoned from the instant the current in phase $a$ reaches its maximum value.

The mmf space waves produced by currents $I_a$, $I_b$, and $I_c$ are identical waves stationary in space, cosinusoidally distributed about the axes of their respective phases and displaced from each other by 120 electrical space degrees or $2\pi/3$ electrical radians. These waves pulsate in time with the currents in their respective phases. Each wave may be resolved into a fundamental mmf space wave and odd-harmonic mmf space waves, cosinusoidally distributed about the axis of the phase, and pulsating in time with the current in the phase.

Let $\gamma$ be the angular displacement in electrical radians of any point $P$ (see Fig. 3, Chapter VII) on the armature surface from the axis of phase $a$, measured in the direction $abc$, which is the positive direction of rotation of the rotor. Let $a_a$, $a_b$, and $a_c$ be the instantaneous mmf's at $P$ produced by currents in phases $a$, $b$, and $c$, respectively, the total instantaneous mmf at $P$ being $a = a_a + a_b + a_c$. Then,

$$a_a = \{F_1 \cos\gamma + F_3 \cos(3\gamma) + F_5 \cos(5\gamma) + \cdots\} \cos t \qquad \text{[B–1]}$$

$$\begin{aligned} a_b = \{&F_1 \cos(\gamma - \tfrac{2}{3}\pi) + F_3 \cos[3(\gamma - \tfrac{2}{3}\pi)] \\ &+ F_5 \cos[5(\gamma - \tfrac{2}{3}\pi)] + \cdots\} \cos(t - \tfrac{2}{3}\pi) \end{aligned} \qquad \text{[B–2]}$$

$$\begin{aligned} a_c = \{&F_1 \cos(\gamma + \tfrac{2}{3}\pi) + F_3 \cos[3(\gamma + \tfrac{2}{3}\pi)] \\ &+ F_5 \cos[5(\gamma + \tfrac{2}{3}\pi)] + \cdots\} \cos(t + \tfrac{2}{3}\pi) \end{aligned} \qquad \text{[B–3]}$$

where $F_1$, $F_3$, $F_5, \cdots$ indicate the amplitudes of the first (fundamental), third, fifth, $\cdots$ harmonic mmf space waves, respectively. The terms in braces $\{\}$ are constant for a constant current $I_m$ and a given value of $\gamma$; they are the mmf waves stationary in space. When multiplied by $\cos t$, $\cos(t - 2\pi/3)$, and $\cos(t + 2\pi/3)$, respectively, the instantaneous mmf's at $P$ produced by currents $I_a$, $I_b$, and $I_c$ are obtained. Equations [B–1]–[B–3] are equations of mmf waves stationary in space but pulsating in time at fundamental frequency.

Let numerical subscript with $a_a$, $a_b$, and $a_c$ indicate the harmonic order of the space waves; then, the instantaneous mmf at $P$ resulting from the three fundamental mmf space waves of amplitude $F_1$, pulsating in time with their respective currents, is $a_1 = a_{a1} + a_{b1} + a_{c1}$, where

$$a_{a1} = [F_1 \cos\gamma] \cos t = \frac{F_1}{2} \cos(\gamma - t) + \frac{F_1}{2} \cos(\gamma + t)^* \qquad \text{[B–4]}$$

$$\begin{aligned} a_{b1} &= \left[F_1 \cos\left(\gamma - \frac{2\pi}{3}\right)\right] \cos\left(t - \frac{2\pi}{3}\right) \\ &= \frac{F_1}{2} \cos(\gamma - t) + \frac{F_1}{2} \cos\left(\gamma + t + \frac{2\pi}{3}\right) \end{aligned} \qquad \text{[B–5]}$$

$$\begin{aligned} a_{c1} &= \left[F_1 \cos\left(\gamma + \frac{2\pi}{3}\right)\right] \cos\left(t + \frac{2\pi}{3}\right) \\ &= \frac{F_1}{2} \cos(\gamma - t) + \frac{F_1}{2} \cos\left(\gamma + t - \frac{2\pi}{3}\right) \end{aligned} \qquad \text{[B–6]}$$

The stationary pulsating wave in each of the above equations is expressed by the terms on the right-hand sides of the second equality signs as the sum of two traveling waves of one-half amplitude, the first rotating forward and the second backward at synchronous speed. To verify this, let point $P$ move forward at synchronous speed, the increase in $\gamma$ in radians equaling the increase in $t$ in radians; then,

---

* $\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$.

$\cos (\gamma - t)$ is constant and the ordinates of the waves represented by the first terms on the right-hand sides of [B–4]–[B–6] remain constant in magnitude at $P$ as $P$ rotates forward at synchronous speed. Similarly, let point $P$ move backward at synchronous speed, the decrease in $\gamma$ being equal numerically to the increase in $t$; then, $(\gamma + t)$, $(\gamma + t + 2\pi/3)$, and $(\gamma + t - 2\pi/3)$ will remain constant, and the ordinates of the waves given by the second terms on the right-hand sides of [B–4]–[B–6] remain constant in magnitude at $P$ as $P$ rotates backward at synchronous speed. As these three backward-rotating waves are equal in amplitude and displaced from each other by 120°, their sum is zero (the sum of the cosines of three angles which differ by 120° is zero), and there is no resultant backward-rotating fundamental mmf space wave. Addition of the above equations gives

$$a_1 = a_{a1} + a_{b1} + a_{c1} = \tfrac{3}{2}F_1 \cos (\gamma - t) = A_1 \cos (\gamma - t) \qquad \text{[B–7]}$$

From [B–7], the fundamental mmf space wave rotates forward at synchronous speed relative to the stator. Its crest $A_1$ is $\frac{3}{2}$ that of the fundamental component of any one of the pulsating mmf space waves given by [B–1]–[B–3].

The three *third harmonic space waves* in [B–1]–[B–3] are identical, because

$$\cos \left[3\left(\gamma \pm \frac{2\pi}{3}\right)\right] = \cos (3\gamma \pm 2\pi) = \cos 3\gamma$$

The instantaneous mmf $a_3$ at $P$ resulting from the three third harmonic mmf space waves of amplitude $F_3$, pulsating at fundamental frequency, is

$$\begin{aligned} a_3 &= a_{a3} + a_{b3} + a_{c3} \\ &= F_3 \cos 3\gamma \left[\cos t + \cos \left(t - \frac{2\pi}{3}\right) + \cos \left(t + \frac{2\pi}{3}\right)\right] = 0 \qquad \text{[B–8]} \end{aligned}$$

From [B–8], there is no resultant third harmonic mmf space wave.

The instantaneous mmf at $P$ resulting from the three pulsating *fifth harmonic* mmf space waves of amplitude $F_5$ is $a_5 = a_{a5} + a_{b5} + a_{c5}$, where

$$a_{a5} = F_5 \cos 5\gamma \cos t = \frac{F_5}{2} \cos (5\gamma - t) + \frac{F_5}{2} \cos (5\gamma + t)$$

$$\begin{aligned} a_{b5} &= F_5 \cos \left[5\left(\gamma - \frac{2\pi}{3}\right)\right] \cos \left(t - \frac{2\pi}{3}\right) \\ &= \frac{F_5}{2} \cos \left(5\gamma - t - \frac{2\pi}{3}\right) + \frac{F_5}{2} \cos (5\gamma + t) \end{aligned}$$

$$a_{c5} = F_5 \cos \left[ 5\left(\gamma + \frac{2\pi}{3}\right) \right] \cos \left(t + \frac{2\pi}{3}\right)$$

$$= \frac{F_5}{2} \cos \left(5\gamma - t + \frac{2\pi}{3}\right) + \frac{F_5}{2} \cos (5\gamma + t)$$

The first terms on the right-hand sides of the above equations (which represent forward-rotating waves) add to zero; therefore,

$$a_5 = a_{a5} + a_{b5} + a_{c5} = \tfrac{3}{2}F_5 \cos (5\gamma + t) = A_5 \cos (5\gamma + t) \quad \text{[B-9]}$$

From [B-9], the fifth harmonic mmf space wave, with an amplitude $A_5$ equal to $\frac{3}{2}$ that of the fifth harmonic component of any one of the stationary pulsating mmf waves given by [B-1]–[B-3], rotates backward at $\frac{1}{5}$ synchronous speed.

By following a similar procedure, it can be shown that the seventh harmonic mmf space wave rotates forward at $\frac{1}{7}$ synchronous speed; the ninth vanishes; the eleventh rotates backward at $\frac{1}{11}$ synchronous speed; the thirteenth rotates forward at $\frac{1}{13}$ synchronous speed, etc.

With positive-sequence currents only in the armature, the instantaneous mmf $a$ at $P$ resulting from these currents is

$$\begin{aligned} a = {} & A_1 \cos (\gamma - t) + A_5 \cos (5\gamma + t) + A_7 \cos (7\gamma - t) \\ & + A_{11} \cos (11\gamma + t) + A_{13} \cos (13\gamma - t) + \cdots \end{aligned} \quad \text{[B-10]}$$

Equation [B-10] is repeated in [2], Chapter VII, and [4] and [5] for $a_d$ and $a_q$ are written by analogy from it.

# APPENDIX C

## RECIPROCALS OF EQUATIONS FOR CIRCLES AND STRAIGHT LINES[1]

**To Prove:** The terminus of the vector denoted by the complex expression

$$V = x + jy = \frac{1}{A + B\underline{/\phi}} \qquad \text{[C–1]}$$

describes a circle under the following conditions:

1. $\phi$ varies from 0° to 360°; $A$ and $B$ are constant;
2. $B$ varies from 0 to $\infty$; $A$ and $\phi$ are constant;

where $A$ and $B$ are real and positive.

*Proof:* If $B\underline{/\phi}$ is expressed in the complex form, [C–1] becomes

$$x + jy = \frac{1}{A + B\cos\phi + jB\sin\phi} \qquad \text{[C–2]}$$

If [C–2] is cleared of fractions,

$$Ax + Bx\cos\phi - By\sin\phi + j(Ay + By\cos\phi + Bx\sin\phi) = 1 + j0$$

When reals and imaginaries are equated,

$$Ax + Bx\cos\phi - By\sin\phi = 1 \qquad \text{[C–3]}$$

$$Ay + By\cos\phi + Bx\sin\phi = 0 \qquad \text{[C–4]}$$

**Case 1.** $\phi$ varies from 0° to 360°. $A$ and $B$ are constant.

$$\rho = A + B\underline{/\phi}$$

is then the equation of a circle in polar coordinates.

Multiplying [C–3] by $x$ and [C–4] by $y$ and adding, there results

$$A(x^2 + y^2) + B(x^2 + y^2)\cos\phi = x$$

$$\cos\phi = \frac{x - A(x^2 + y^2)}{B(x^2 + y^2)} \qquad \text{[C–5]}$$

If [C–3] and [C–4] are multiplied by $y$ and $x$, respectively, and their difference taken,

$$B(x^2 + y^2)\sin\phi = -y$$

$$\sin\phi = \frac{-y}{B(x^2 + y^2)} \qquad [C\text{–}6]$$

$$\sin^2\phi + \cos^2\phi = 1 \qquad [C\text{–}7]$$

Substituting [C–5] and [C–6] in [C–7], there results

$$\frac{(x^2 + y^2) - 2A(x^2 + y^2)x + A^2(x^2 + y^2)^2}{B^2(x^2 + y^2)^2} = 1$$

$$1 - 2Ax + A^2(x^2 + y^2) = B^2(x^2 + y^2)$$

$$x^2 - \frac{2Ax}{A^2 - B^2} + \frac{A^2}{(A^2 - B^2)^2} + y^2 = \frac{A^2}{(A^2 - B^2)^2} - \frac{A^2 - B^2}{(A^2 - B^2)^2}$$

$$\left[x - \frac{A}{A^2 - B^2}\right]^2 + y^2 = \left[\frac{B}{(A^2 - B^2)}\right]^2 \qquad [C\text{–}8]$$

Equation [C–8] is the equation of a circle in rectangular coordinates. As $\phi$ varies from 0° to 360°, the terminus of the vector $V = x + jy$ will move completely around the circle.

If $r$ and $C$ denote radius and center of the circle, respectively,

$$r = \left|\frac{B}{A^2 - B^2}\right| = \left|\frac{B}{B^2 - A^2}\right| \qquad [C\text{–}9]$$

$$C = \frac{A}{A^2 - B^2} + j0 = \frac{A}{A^2 - B^2}\,\underline{/0^\circ} \qquad [C\text{–}10]$$

It is of interest to note that the same equations apply for $r$ and $C$ if $V = x + jy = 1/(A - B\underline{/\phi})$.

From [C–9] and [C–10], $V$ can be written as the equation of a circle in polar coordinates:

$$V = C + r\underline{/\theta} = \frac{1}{A + B\underline{/\phi}} = \frac{A}{A^2 - B^2}\,\underline{/0^\circ} + \left|\frac{B}{A^2 - B^2}\right|\underline{/\theta} \qquad [C\text{–}11]$$

where $\theta$ has all values between 0° and 360°. $\theta$ is not the same as $\phi$ but, if $\phi$ covers the range from 0° to 360°, $\theta$ will also cover this range.

**Case 2.** $B$ varies from 0 to $\infty$. $A$ and $\phi$ are constant.

$$\rho = A + B\underline{/\phi}$$

is then the equation of a straight line.

If $B$ is eliminated from [C–3] and [C–4],

$$Ax(y\cos\phi + x\sin\phi) - Ay(x\cos\phi - y\sin\phi) = y\cos\phi + x\sin\phi$$

$$A\sin\phi(x^2 + y^2) - x\sin\phi - y\cos\phi = 0$$

$$x^2 - \frac{x}{A} + y^2 - \frac{y}{A}\cot\phi = 0$$

$$x^2 - \frac{1}{A}x + \frac{1}{4A^2} + y^2 - \frac{\cot\phi}{A}y + \frac{\cot^2\phi}{4A^2} = \frac{1 + \cot^2\phi}{4A^2}$$

$$\left(x - \frac{1}{2A}\right)^2 + \left(y - \frac{\cot\phi}{2A}\right)^2 = \left(\frac{\csc\phi}{2A}\right)^2 \qquad \text{[C–12]}$$

Equation [C–12] is the equation of a circle. If $C'$ and $r'$ denote center and radius of this circle, respectively,

$$C' = \frac{1}{2A} + j\frac{\cot\phi}{2A} = \frac{1}{2A\sin\phi}\underline{/\tan^{-1}\cot\phi}$$

$$= \frac{1}{2A\sin\phi}\underline{/90^\circ - \phi} \qquad \text{[C–13]}$$

$$r' = \left|\frac{\csc\phi}{2A}\right| = \left|\frac{1}{2A\sin\phi}\right| \qquad \text{[C–14]}$$

**Case 2 (a).** Let

$$V = x + jy = \frac{1}{A - B\underline{/\phi}} \qquad \text{[C–15]}$$

It can readily be shown that the equations for $C'$ and $r'$ given by [C–13] and [C–14] apply to this case also.

From [C–13] and [C–14], $V$ given by either [C–1] or [C–15] can be written as the equation of a circle in polar coordinates.

$$V = C' + r'\underline{/\theta'} = \frac{1}{2A\sin\phi}\underline{/90^\circ - \phi} + \left|\frac{1}{2A\sin\phi}\right|\underline{/\theta'} \qquad \text{[C–16]}$$

where $\theta'$ has all values between 0° and 360° as $B$ varies from $-\infty$ through 0 to $+\infty$. As $B$ was eliminated in developing [C–12], the restriction that $B$ must be positive has disappeared. It is necessary to consider the original equation, [C–1] or [C–15], to impose this restriction.

There are the relations:

$$\cot\alpha = \cot(\alpha + 180^\circ)$$

$$|\sin\alpha| = |\sin(\alpha + 180^\circ)|$$

From the above equations, it follows that the center and radius of the circle given by [C–13] and [C–14] will be the same for $\phi = \alpha$ as for $\phi = (180° + \alpha)$. The complete circle will apply for $\phi = \alpha$ and for $\phi = (180° + \alpha)$. The portion which applies to each can be determined from a consideration of the original equation for $V$, given by [C–1] or [C–15].

Consider $V$ given by [C–15]. If $\phi$ lies between 0° and 180°, the $j$ component of $V$ will be positive and the arc above the $x$-axis will apply. If $\phi$ lies between 180° and 360°, the $j$ component of $V$ will be negative and the arc below the $x$-axis will apply. If $\phi = 0$ or 180°, the circle becomes a straight line which coincides with the $x$-axis. If $\phi = 180°$, the denominator in [C–15] becomes $A + B$ and $V$ extends from 0 (when $B = \infty$) to $1/A$ (when $B = 0$). The rest of the line, which extends from 0 to $-\infty$ and from $+\infty$ to $1/A$, is the locus of $V$ when $\phi = 0$.

If $V$ given by [C–1] is considered, the conditions will be the reverse of those for $V$ given by [C–15]. For example: When $V$ is given by [C–1], the line between $1/A$ and zero is the locus of $V$ when $\phi = 0$.

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